Pensieve header: The Logos in the vcw order, following Roland.

## Implementing $qg_l$

```
ϵ /: ϵ^2 = 0; ϵ /: ϵ^3 = 0;
PBWRule = {v → 1, c → 2, w → 3};
B[U@c, U@w] = -(B[U@w, U@c] = U@w);
B[U@v, U@c] = -(B[U@c, U@v] = U@v);
B[U@w, U@v] = -(B[U@v, U@w] = (t - 1) U[] + ϵ (t + 1) U@c);
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[]; UU[l_, r___] := U[l] ** UU[r];
U_i_[ℰ_] := ℰ /. {b → b_i, t → t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
```

```
B[x_, x_] = 0;
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, y_] := x ** y - y ** x;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := If[x ≺ y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy__] := If[x ≤ y1, U[x, y1, yy], (U@x ** U@y1) ** U@yy];
U[xx__, xn_] ** U[yy__] := U@xx ** (U@xn ** U@yy);
```

```
U[l___, x_^n_, r___] := U[l, Sequence @@ Table[x, {n}], r];
U[l___, 1, r___] := U[l, r];
```

```
ToDegree[n_][ℰ_] := Simp[ℰ] /. a_. x_U :→ Normal[Series[a, {ℏ, 0, n}]] * x
```

```
LBasis[n_Integer] := LBasis[Range[n]];
LBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, v_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
       Table[{U@c_i, U@v_i, U@w_i, ϵ U@c_i, ϵ U@v_i, ϵ U@w_i}, {i, S}],
       Table[{U[v_i, w_j], ϵ U[v_i, w_j],
         ϵ U @@ Sort@{c_i, c_j}, ϵ U[v_j, c_i], ϵ U[c_i, w_j]}, {i, S}, {j, S}],
       Table[{ϵ U[v_j, c_i, w_k], ϵ U @@ Sort@{v_i, v_j, w_k}, ϵ U @@ Sort@{v_i, w_j, w_k}},
        {i, S}, {j, S}, {k, S}],
       Table[ϵ U @@ Sort@{v_i, v_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
BLBasis[n_Integer] := BLBasis[Range[n]];
BLBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, v_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
       Table[{U@c_i, ϵ U@c_i}, {i, S}],
       Table[{U[v_i, w_j], ϵ U[v_i, w_j], ϵ U @@ Sort@{c_i, c_j}}, {i, S}, {j, S}],
       Table[{ϵ U[v_j, c_i, w_k]}, {i, S}, {j, S}, {k, S}],
       Table[ϵ U @@ Sort@{v_i, v_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
O[n_, poly_, specs___] := Module[{vs, us},
  vs = Join @@ (First /@ {specs});
  us = Join @@ ({specs} /. (l_ → s_) :→ (l /. x_i_ :→ x_s));
  Total[
   CoefficientRules[Normal@Series[poly, {ħ, 0, n}], vs] /. (p_ → c_) :→ c UU @@ (us^p)]]
```

## Testing $g_l$

```
B[U@v1, U@w1]
```

$(-1 + t_1)\, U[\,] + \epsilon\, (1 + t_1)\, U[c_1]$

```
LBasis[2]
```

{U[], ϵ U[], U[c1], U[c2], U[v1], U[v2], U[w1], U[w2], ϵ U[c1], ϵ U[c2], ϵ U[v1], ϵ U[v2], ϵ U[w1], ϵ U[w2], U[v1, w1], U[v1, w2], U[v2, w1], U[v2, w2], ϵ U[c1, c1], ϵ U[c1, c2], ϵ U[c1, w1], ϵ U[c1, w2], ϵ U[c2, c2], ϵ U[c2, w1], ϵ U[c2, w2], ϵ U[v1, c1], ϵ U[v1, c2], ϵ U[v1, w1], ϵ U[v1, w2], ϵ U[v2, c1], ϵ U[v2, c2], ϵ U[v2, w1], ϵ U[v2, w2], ϵ U[v1, c1, w1], ϵ U[v1, c1, w2], ϵ U[v1, c2, w1], ϵ U[v1, c2, w2], ϵ U[v1, v1, w1], ϵ U[v1, v1, w2], ϵ U[v1, v2, w1], ϵ U[v1, v2, w2], ϵ U[v1, w1, w1], ϵ U[v1, w1, w2], ϵ U[v1, w2, w2], ϵ U[v2, c1, w1], ϵ U[v2, c1, w2], ϵ U[v2, c2, w1], ϵ U[v2, c2, w2], ϵ U[v2, v2, w1], ϵ U[v2, v2, w2], ϵ U[v2, w1, w1], ϵ U[v2, w1, w2], ϵ U[v2, w2, w2], ϵ U[v1, v1, w1, w1], ϵ U[v1, v1, w1, w2], ϵ U[v1, v1, w2, w2], ϵ U[v1, v2, w1, w1], ϵ U[v1, v2, w1, w2], ϵ U[v1, v2, w2, w2], ϵ U[v2, v2, w1, w1], ϵ U[v2, v2, w1, w2], ϵ U[v2, v2, w2, w2]}

```
bas = LBasis[2];
Table[B[x, y] + B[y, x], {x, bas}, {y, bas}] // Flatten // Union
```

{0}

```
bas = LBasis[2]; Timing[
 Table[
   {x, y, z} = xyz;
   Simp[B[B[x, y], z] + B[B[y, z], x] + B[B[z, x], y]],
   {xyz, Subsets[bas, {3}]}
  ] // Flatten // Union
]
```

{47.4531, {0}}

## Testing bi-local exponential relations in $g_0$

1. c Relations.

$\mathbb{O}\left[3,\ e^{\hbar\,(\gamma\, c_0+\beta\, v_0)},\ \{v_0,\ c_0\}\to 0\right]$

$$U[\,] + \gamma\hbar\, U[c_0] + \beta\hbar\, U[v_0] + \frac{1}{2}\gamma^2\hbar^2\, U[c_0, c_0] + \beta\gamma\hbar^2\, U[v_0, c_0] + \frac{1}{2}\beta^2\hbar^2\, U[v_0, v_0] + \frac{1}{6}\gamma^3\hbar^3\, U[c_0, c_0, c_0] + \frac{1}{2}\beta\gamma^2\hbar^3\, U[v_0, c_0, c_0] + \frac{1}{2}\beta^2\gamma\hbar^3\, U[v_0, v_0, c_0] + \frac{1}{6}\beta^3\hbar^3\, U[v_0, v_0, v_0]$$

$\mathbb{O}\left[3,\ e^{\hbar\,\left(\gamma\, c_0+e^{-\hbar\gamma}\beta\, v_0\right)},\ \{c_0,\ v_0\}\to 0\right]$

$$U[\,] + \gamma\hbar\, U[c_0] + \left(\beta\hbar - \beta\gamma\hbar^2 + \frac{1}{2}\beta\gamma^2\hbar^3\right) U[v_0] + \frac{1}{2}\gamma^2\hbar^2\, U[c_0, c_0] + \left(\beta\gamma\hbar^2 - \beta\gamma^2\hbar^3\right)\left(U[v_0] + U[v_0, c_0]\right) + \left(\frac{\beta^2\hbar^2}{2} - \beta^2\gamma\hbar^3\right) U[v_0, v_0] + \frac{1}{6}\gamma^3\hbar^3\, U[c_0, c_0, c_0] + \frac{1}{2}\beta\gamma^2\hbar^3\left(U[v_0] + 2\, U[v_0, c_0] + U[v_0, c_0, c_0]\right) + \frac{1}{2}\beta^2\gamma\hbar^3\left(2\, U[v_0, v_0] + U[v_0, v_0, c_0]\right) + \frac{1}{6}\beta^3\hbar^3\, U[v_0, v_0, v_0]$$

$\text{With}\left[\{n = 7\},\ \text{Simp}\left[\mathbb{O}\left[n,\ e^{\hbar\,(\gamma\, c_0+\beta\, v_0)},\ \{v_0,\ c_0\}\to 0\right] == \mathbb{O}\left[n,\ e^{\hbar\,\left(\gamma\, c_0+e^{-\hbar\gamma}\beta\, v_0\right)},\ \{c_0,\ v_0\}\to 0\right]\right]\right]$

True

$\text{With}\left[\{n = 7\},\ \text{Simp}\left[\mathbb{O}\left[n,\ e^{\hbar\,(\gamma\, c_0+\hbar\,\beta\, w_0)},\ \{w_0,\ c_0\}\to 0\right] == \mathbb{O}\left[n,\ e^{\hbar\,\left(\gamma\, c_0+\hbar\, e^{\hbar\gamma}\beta\, w_0\right)},\ \{c_0,\ w_0\}\to 0\right]\right]\right]$

True

2. v-w Relations.

$$\Lambda[\alpha\_, \beta\_, \delta\_, \mu\_, v\_, c\_, w\_, t\_] := -\frac{1+t}{4}\left(4\, v\, c\, w\, \delta^2\mu^2 + \delta\,(1+\mu)\left(w^2\alpha^2 + v^2\beta^2\right) + v^2 w^2\delta^3\,(1+3\,\mu) + (1-t)\left(2\,(\alpha\beta+\delta\mu)^2 - \alpha^2\beta^2\right) + 2\left(\alpha\beta + 2\,\delta\mu + v\, w\,\delta^2\,(1+2\,\mu) + 2\, c\,\delta\mu^2\right)(w\,\alpha + v\,\beta) + 4\left(c\,\mu^2 + v\, w\,\delta\,(1+\mu)\right)(\alpha\beta+\delta\mu)\right);$$

```
With[{n = 7}, ToDegree[n][
  (𝕆[n, ⅇ^(ℏ (α w0+β v0+δ v0 w0)), {w0, v0} → 0]) -
   𝕆[n, μ^-1 (1 + ϵ μ^-4 Λ[ℏ α, ℏ β, ℏ δ, μ, v0, c0, w0, t0]) ⅇ^((ℏ^2 (1-t0) α β+ℏ α w0+ℏ β v0+ℏ δ v0 w0)/μ) /.
     μ → 1 - (1 - t0) ℏ δ, {v0, c0, w0} → 0]
 ]]
```

0

```
FullSimplify[Λ[α, β, δ, μ, v, c, w, t]]
```

$$\frac{1}{4}(-1-t)\Big(4\,c\,v\,w\,\delta^2\mu^2+(w^2\alpha^2+v^2\beta^2)\,\delta\,(1+\mu)+v^2w^2\delta^3(1+3\mu)+4\,(\alpha\beta+\delta\mu)\,(c\,\mu^2+v\,w\,\delta\,(1+\mu))+(1-t)\,(-\alpha^2\beta^2+2\,(\alpha\beta+\delta\mu)^2)+2\,(w\,\alpha+v\,\beta)\,(\alpha\beta+\delta\,(v\,w\,\delta\,(1+2\mu)+2\mu\,(1+c\,\mu)))\Big)$$