Pensieve header: The Logos in the ucw order (automatic but messy degrees).

## Implementing $g_1$

```
ϵ /: ϵ^2 = 0;
PBWRule = {u → 1, c → 2, w → 3};
Clear[Deg];
Evaluate[Deg /@ {b, c, u, w, ϵ}] = {1, 0, 1, 0, 1};
Deg[x_] := Deg[x];
B[U@c, U@w] = -(B[U@w, U@c] = U@w);
B[U@u, U@c] = -(B[U@c, U@u] = U@u);
B[U@w, U@u] = -(B[U@u, U@w] = b U[] - 2 ϵ U@c);
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[]; UU[l_, r___] := U[l] ** UU[r];
U_i_[ℰ_] := ℰ /. {b → b_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
```

```
B[x_, x_] = 0;
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, y_] := x ** y - y ** x;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := If[x < y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy__] := If[x ≤ y1, U[x, y1, yy], (U@x ** U@y1) ** U@yy];
U[xx__, xn_] ** U[yy__] := U@xx ** (U@xn ** U@yy);
```

```
U[l___, x_^n_, r___] := U[l, Sequence @@ Table[x, {n}], r];
U[l___, 1, r___] := U[l, r];
```

```
bc_i_ := b_i U[] - ϵ U[c_i];
a_i_,j_ := bc_i ** U@c_j + U@u_i ** U@w_j;
```

```
UExp[ℰ_, n_] := Module[{t = U[], k}, U[] + Sum[(t = t ** ℰ)/k!, {k, n}]] // Simp
```

```
ToDegree[n_][ℰ_] := Normal[
  Simp[ℰ] /. {ϵ → ℏ ϵ, b_i_ :→ ℏ b_i, b → ℏ b, x_U :→ ℏ^Count[x,u|u_] x} /.
   a_. x_U :→ Series[a, {ℏ, 0, n}] * x]
```

```
LBasis[n_Integer] := LBasis[Range[n]];
LBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, u_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
        Table[{U@c_i, U@u_i, U@w_i, ϵ U@c_i, ϵ U@u_i, ϵ U@w_i}, {i, S}],
        Table[{U[u_i, w_j], ϵ U[u_i, w_j],
          ϵ U @@ Sort@{c_i, c_j}, ϵ U[u_j, c_i], ϵ U[c_i, w_j]}, {i, S}, {j, S}],
        Table[{ϵ U[u_j, c_i, w_k], ϵ U @@ Sort@{u_i, u_j, w_k}, ϵ U @@ Sort@{u_i, w_j, w_k}},
         {i, S}, {j, S}, {k, S}],
        Table[ϵ U @@ Sort@{u_i, u_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
BLBasis[n_Integer] := BLBasis[Range[n]];
BLBasis[S_] := DeleteCases[0]@
  Module[{i, j, k, l}, SortBy[(# /. {ϵ → 2, c_ → 2, u_ → 2, w_ → 2, U → Times}) &][
    Union@Flatten[{{U[], ϵ U[]},
        Table[{U@c_i, ϵ U@c_i}, {i, S}],
        Table[{U[u_i, w_j], ϵ U[u_i, w_j], ϵ U @@ Sort@{c_i, c_j}}, {i, S}, {j, S}],
        Table[{ϵ U[u_j, c_i, w_k]}, {i, S}, {j, S}, {k, S}],
        Table[ϵ U @@ Sort@{u_i, u_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}] }]
   ]]
```

```
𝕆[n_, poly_, specs___] := Module[{vs, us},
  vs = Join @@ (First /@ {specs});
  us = Join @@ ({specs} /. (l_ → s_) :→ (l /. x_i_ :→ x_s));
  Total[
   CoefficientRules[
     Normal@
      Series[poly /. {ϵ → ℏ ϵ, b_i_ :→ ℏ b_i} /. Thread[vs → ℏ^Deg/@vs vs], {ℏ, 0, n}], vs
     ] /. (p_ → c_) :→ c UU @@ (us^p)
   ]
  ]
```

## Testing $g_I$

```
Table[x → B[U@x, a1,2], {x, {c1, c2, u1, u2, w1, w2, c3, u3, w3}}] // Column
```

$c_1 \to U[u_1, w_2]$
$c_2 \to -U[u_1, w_2]$
$u_1 \to \in U[u_1, c_2]$
$u_2 \to b_2 U[u_1] - b_1 U[u_2] - 2 \in U[u_1, c_2] + \in U[u_2, c_1]$
$w_1 \to -b_1 U[w_2] + 2 \in U[c_1, w_2] - \in U[c_2, w_1]$
$w_2 \to b_1 U[w_2] - \in U[c_1, w_2]$
$c_3 \to 0$
$u_3 \to 0$
$w_3 \to 0$

```
B[a1,3, a2,3] + B[a1,2, a2,3] + B[a1,2, a1,3]
```

0

```
{t1 = B[a1,2, a1,3], t2 = ∈ U@c3 ** a1,2 - ∈ U@c2 ** a1,3, t1 - t2} // Expand // Column
```

$-\in U[u_1, c_2, w_3] + \in U[u_1, c_3, w_2]$
$-\in U[u_1, c_2, w_3] + \in U[u_1, c_3, w_2]$
0

```
{t1 = B[a1,3, a2,3], t2 = bc2 ** a1,3 - bc1 ** a2,3, t1 - t2} // Expand // Column
```

$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] - \in U[u_1, c_2, w_3] + \in U[u_2, c_1, w_3]$
$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] - \in U[u_1, c_2, w_3] + \in U[u_2, c_1, w_3]$
0

```
{t1 = a1,3 ** a2,4 - a1,4 ** a2,3,
  t2 = a1,3 ** bc2 ** U@c4 - a2,3 ** bc1 ** U@c4 - a1,4 ** bc2 ** U@c3 + a2,4 ** bc1 ** U@c3,
  t1 - t2
 } // Column
```

$-b_2 U[u_1, c_3, w_4] + b_2 U[u_1, c_4, w_3] + b_1 U[u_2, c_3, w_4] - b_1 U[u_2, c_4, w_3] + \in U[u_1, c_2, c_3, w_4] - \in U[u_1, c_2, c_4, w_3] - \in U[u_2, c_1, c_3, w_4] + \in U[u_2, c_1, c_4, w_3]$
$-b_2 U[u_1, c_3, w_4] + b_2 U[u_1, c_4, w_3] + b_1 U[u_2, c_3, w_4] - b_1 U[u_2, c_4, w_3] + \in U[u_1, c_2, c_3, w_4] - \in U[u_1, c_2, c_4, w_3] - \in U[u_2, c_1, c_3, w_4] + \in U[u_2, c_1, c_4, w_3]$
0

```
LBasis[2]
```

$\{U[], \in U[], U[c_1], U[c_2], U[u_1], U[u_2], U[w_1], U[w_2], \in U[c_1], \in U[c_2], \in U[u_1], \in U[u_2], \in U[w_1], \in U[w_2], U[u_1, w_1], U[u_1, w_2], U[u_2, w_1], U[u_2, w_2], \in U[c_1, c_1], \in U[c_1, c_2], \in U[c_1, w_1], \in U[c_1, w_2], \in U[c_2, c_2], \in U[c_2, w_1], \in U[c_2, w_2], \in U[u_1, c_1], \in U[u_1, c_2], \in U[u_1, w_1], \in U[u_1, w_2], \in U[u_2, c_1], \in U[u_2, c_2], \in U[u_2, w_1], \in U[u_2, w_2], \in U[u_1, c_1, w_1], \in U[u_1, c_1, w_2], \in U[u_1, c_2, w_1], \in U[u_1, c_2, w_2], \in U[u_1, u_1, w_1], \in U[u_1, u_1, w_2], \in U[u_1, u_2, w_1], \in U[u_1, u_2, w_2], \in U[u_1, w_1, w_1], \in U[u_1, w_1, w_2], \in U[u_1, w_2, w_2], \in U[u_2, c_1, w_1], \in U[u_2, c_1, w_2], \in U[u_2, c_2, w_1], \in U[u_2, c_2, w_2], \in U[u_2, u_2, w_1], \in U[u_2, u_2, w_2], \in U[u_2, w_1, w_1], \in U[u_2, w_1, w_2], \in U[u_2, w_2, w_2], \in U[u_1, u_1, w_1, w_1], \in U[u_1, u_1, w_1, w_2], \in U[u_1, u_1, w_2, w_2], \in U[u_1, u_2, w_1, w_1], \in U[u_1, u_2, w_1, w_2], \in U[u_1, u_2, w_2, w_2], \in U[u_2, u_2, w_1, w_1], \in U[u_2, u_2, w_1, w_2], \in U[u_2, u_2, w_2, w_2]\}$

```
bas = LBasis[2];
Table[B[x, y] + B[y, x], {x, bas}, {y, bas}] // Flatten // Union
```

{0}

```
bas = LBasis[2]; Timing[
 Table[
   {x, y, z} = xyz;
   Simp[B[B[x, y], z] + B[B[y, z], x] + B[B[z, x], y]],
   {xyz, Subsets[bas, {3}]}
  ] // Flatten // Union
]
```

{29.6406, {0}}

## Infinitesimal Spinner Relations

$s_{i\_} := \frac{b_i}{2} U[] - \epsilon\, U[c_i]$

$a_{1,2}$

$b_1 U[c_2] - \epsilon\, U[c_1, c_2] + U[u_1, w_2]$

$B[a_{1,2}, s_1]$

$\epsilon\, U[u_1, w_2]$

$B[a_{1,2}, s_2]$

$-\epsilon\, U[u_1, w_2]$

$B[a_{1,2}, s_1 + s_2]$ // Simplify

0

$B[a_{1,3}, a_{2,3}]$

$b_2 U[u_1, w_3] - b_1 U[u_2, w_3] - \epsilon\, U[u_1, c_2, w_3] + \epsilon\, U[u_2, c_1, w_3]$

Simplify$[b_1 UU[u_1, w_3] - b_1 UU[u_1, w_3] + \epsilon\, UU[c_1, u_1, w_3] - \epsilon\, UU[u_1, c_1, w_3]]$

$\epsilon\, U[u_1, w_3]$

## Testing bi-local exponential relations in $g_0$

1. The Yang-Baxter Element.
2. c Relations.

```
B[U[c], U[u, w]]
```

0

```
𝕆[3, ℯ^(ℏ γ c0+β u0), {u0, c0} → 1]
```

$$U[\,] + \gamma\hbar\, U[c_1] + \beta\hbar\, U[u_1] + \frac{1}{2}\gamma^2\hbar^2 U[c_1, c_1] + \beta\gamma\hbar^2 U[u_1, c_1] + \frac{1}{2}\beta^2\hbar^2 U[u_1, u_1] + \frac{1}{6}\gamma^3\hbar^3 U[c_1, c_1, c_1] + \frac{1}{2}\beta\gamma^2\hbar^3 U[u_1, c_1, c_1] + \frac{1}{2}\beta^2\gamma\hbar^3 U[u_1, u_1, c_1] + \frac{1}{6}\beta^3\hbar^3 U[u_1, u_1, u_1]$$

```
𝕆[3, ℯ^(ℏ γ c0+ℯ^(-ℏ γ) β u0), {c0, u0} → 1]
```

$$U[\,] + \gamma\hbar\, U[c_1] + \left(\beta\hbar - \beta\gamma\hbar^2 + \frac{1}{2}\beta\gamma^2\hbar^3\right) U[u_1] + \frac{1}{2}\gamma^2\hbar^2 U[c_1, c_1] + \left(\beta\gamma\hbar^2 - \beta\gamma^2\hbar^3\right)(U[u_1] + U[u_1, c_1]) + \left(\frac{\beta^2\hbar^2}{2} - \beta^2\gamma\hbar^3\right) U[u_1, u_1] + \frac{1}{6}\gamma^3\hbar^3 U[c_1, c_1, c_1] + \frac{1}{2}\beta\gamma^2\hbar^3\left(U[u_1] + 2U[u_1, c_1] + U[u_1, c_1, c_1]\right) + \frac{1}{2}\beta^2\gamma\hbar^3\left(2U[u_1, u_1] + U[u_1, u_1, c_1]\right) + \frac{1}{6}\beta^3\hbar^3 U[u_1, u_1, u_1]$$

```
With[{n = 7}, Simp[𝕆[n, ℯ^(ℏ γ c0+β u0), {u0, c0} → 1] == 𝕆[n, ℯ^(ℏ γ c0+ℯ^(-ℏ γ) β u0), {c0, u0} → 1]]]
```

True

```
With[{n = 7}, Simp[𝕆[n, ℯ^(ℏ γ c0+ℏ β w0), {w0, c0} → 1] == 𝕆[n, ℯ^(ℏ γ c0+ℏ ℯ^(ℏ γ) β w0), {c0, w0} → 1]]]
```

True

## 3. $u$, $w$, $e^u$, $e^w$.

```
With[{n = 10},
 Simp[B[U@w, UExp[γ U@u, n]] + (b γ U[] - 2 γ ∈ U[c] + ∈ γ^2 U[u]) ** UExp[γ U@u, n]]]
```

$$\left(\frac{b\gamma^{11}}{3628800} - \frac{\gamma^{11}\in}{362880}\right) U[u, u, u, u, u, u, u, u, u, u] - \frac{\gamma^{11}\in U[u, u, u, u, u, u, u, u, u, u, c]}{1814400} + \frac{\gamma^{12}\in U[u, u, u, u, u, u, u, u, u, u, u]}{3628800}$$

```
With[{n = 7},
 Simp[B[U@u, UExp[γ U@w, n]] - (b γ U[] - 2 γ ∈ U[c] - ∈ γ^2 U[w]) ** UExp[γ U@w, n]]]
```

$$\left(-\frac{b\gamma^{8}}{5040} + \frac{\gamma^{8}\in}{720}\right) U[w, w, w, w, w, w, w] + \frac{\gamma^{8}\in U[c, w, w, w, w, w, w, w]}{2520} + \frac{\gamma^{9}\in U[w, w, w, w, w, w, w, w]}{5040}$$

## 4. $M_{uw}$ Relations.

```
Muw[γ_, n_] := Expand[Sum[γ^k/k! UU[u^k, w^k], {k, 0, n}]]
```

```
Muw[γ, 5]
```

$$U[\,] + \gamma U[u, w] + \frac{1}{2}\gamma^2 U[u, u, w, w] + \frac{1}{6}\gamma^3 U[u, u, u, w, w, w] + \frac{1}{24}\gamma^4 U[u, u, u, u, w, w, w, w] + \frac{1}{120}\gamma^5 U[u, u, u, u, u, w, w, w, w, w]$$

**𝕆[5, $e^{\delta\, u_0 w_0}$, {$u_0$, $w_0$} → 1]**

$$U[\,] + \delta\,\hbar\,U[u_1, w_1] + \frac{1}{2}\delta^2\hbar^2\,U[u_1, u_1, w_1, w_1] + \frac{1}{6}\delta^3\hbar^3\,U[u_1, u_1, u_1, w_1, w_1, w_1] + \frac{1}{24}\delta^4\hbar^4\,U[u_1, u_1, u_1, u_1, w_1, w_1, w_1, w_1] + \frac{1}{120}\delta^5\hbar^5\,U[u_1, u_1, u_1, u_1, u_1, w_1, w_1, w_1, w_1, w_1]$$

```
With[{n = 12}, Muw[γ, n] ** U[u] -
    Expand@((1 - b γ - 2 γ ϵ) U[u] + 2 γ ϵ UU[c, u] - (1 - b γ)^-1 γ^2 ϵ U[u, u, w]) **
     Muw[γ, n]] // Simplify // Expand
```

$$-\left(\left(b\,\gamma^{13}\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(479\,001\,600\,(-1+b\,\gamma)\right)\right) +$$
$$\left(b^2\,\gamma^{14}\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(479\,001\,600\,(-1+b\,\gamma)\right) +$$
$$\left(\gamma^{13}\,\epsilon\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(39\,916\,800\,(-1+b\,\gamma)\right) -$$
$$\left(b\,\gamma^{14}\,\epsilon\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(39\,916\,800\,(-1+b\,\gamma)\right) +$$
$$\left(\gamma^{13}\,\epsilon\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,c,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(239\,500\,800\,(-1+b\,\gamma)\right) -$$
$$\left(b\,\gamma^{14}\,\epsilon\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,c,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(239\,500\,800\,(-1+b\,\gamma)\right) -$$
$$\left(\gamma^{14}\,\epsilon\,U[u,u,u,u,u,u,u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w,w,w,w,w,w]\right)\Big/\left(479\,001\,600\,(-1+b\,\gamma)\right)$$

```
With[{n = 7}, Simplify[B[U[w], Muw[γ, n]] + b γ Muw[γ, n] ** U[w]]]
```

$$2\,\gamma\,\epsilon\,U[c,w] + \gamma^2\,\epsilon\,U[u,w,w] + 2\,\gamma^2\,\epsilon\,U[u,c,w,w] +$$
$$\gamma^3\,\epsilon\,U[u,u,w,w,w] + \gamma^3\,\epsilon\,U[u,u,c,w,w,w] + \frac{1}{2}\gamma^4\,\epsilon\,U[u,u,u,w,w,w,w] +$$
$$\frac{1}{3}\gamma^4\,\epsilon\,U[u,u,u,c,w,w,w,w] + \frac{1}{6}\gamma^5\,\epsilon\,U[u,u,u,u,w,w,w,w,w] +$$
$$\frac{1}{12}\gamma^5\,\epsilon\,U[u,u,u,u,c,w,w,w,w,w] + \frac{1}{24}\gamma^6\,\epsilon\,U[u,u,u,u,u,w,w,w,w,w,w] +$$
$$\frac{1}{60}\gamma^6\,\epsilon\,U[u,u,u,u,u,c,w,w,w,w,w,w] + \frac{1}{120}\gamma^7\,\epsilon\,U[u,u,u,u,u,u,w,w,w,w,w,w,w] +$$
$$\frac{1}{360}\gamma^7\,\epsilon\,U[u,u,u,u,u,u,c,w,w,w,w,w,w,w] +$$
$$\frac{1}{5040}\,b\,\gamma^8\,U[u,u,u,u,u,u,u,w,w,w,w,w,w,w,w]$$

```
With[{n = 7}, Simplify[
  (∂_α Muw[γ[α], n]) - γ'[α]/(1 - b γ[α]) Muw[γ[α], n] ** U[u, w]
 ]]
```

$$\frac{1}{5040\,(-1+b\,\gamma[\alpha])}\gamma[\alpha]\,(10080\,\epsilon\,U[u,c,w]+\gamma[\alpha]\,(5040\,\epsilon\,U[u,u,w,w]+10080\,\epsilon\,U[u,u,c,w,w]+\gamma[\alpha]\,(5040\,\epsilon\,U[u,u,u,w,w,w]+5040\,\epsilon\,U[u,u,u,c,w,w,w]+\gamma[\alpha]\,(2520\,\epsilon\,U[u,u,u,u,w,w,w,w]+1680\,\epsilon\,U[u,u,u,u,c,w,w,w,w]+\gamma[\alpha]\,(840\,\epsilon\,U[u,u,u,u,u,w,w,w,w,w]+420\,\epsilon\,U[u,u,u,u,u,c,w,w,w,w,w]+\gamma[\alpha]\,(210\,\epsilon\,U[u,u,u,u,u,u,w,w,w,w,w,w]+84\,\epsilon\,U[u,u,u,u,u,u,c,w,w,w,w,w,w]+(42\,\epsilon\,U[u,u,u,u,u,u,u,w,w,w,w,w,w,w]+14\,\epsilon\,U[u,u,u,u,u,u,u,c,w,w,w,w,w,w,w]+U[u,u,u,u,u,u,u,u,w,w,w,w,w,w,w,w])\,\gamma[\alpha]))))))\,\gamma'[\alpha]$$

## 5. $M_{wu}$ Relations.

```
Mwu[γ_, n_] := Expand[Sum_{k=0}^{n} γ^k/k! UU[w^k, u^k]]
```

```
With[{n = 7}, {t1 = Simplify[B[U[u], Mwu[δ, n]] - b δ Mwu[δ, n] ** U[u]], t1 // ToDegree[n]}]
```

```
With[{n = 7}, {t1 = Simp[B[U[w], Mwu[δ, n]] + b δ U[w] ** Mwu[δ, n]], t1 // ToDegree[n]}]
```

```
With[{n = 7}, {t1 = Simplify[
    (∂_α Mwu[α δ, n]) - δ/(1 + b α δ) Mwu[α δ, n] ** U[w] ** U[u]
  ], t1 // ToDegree[n]}]
```

```
With[{n = 7}, Simplify[
  (∂_α Mwu[α δ, n]) - δ/(1 + b α δ) Mwu[α δ, n] ** (U[u, w] - b U[])
 ]]
```

## 6. $M_{wu}$ and $M_{uw}$.

```
With[{n = 10, ν = (1 + b δ)^-1}, t1 = Simp[
    -Mwu[δ, n] + (ν (1 - ν^2 b δ^2 ϵ) U[] + 2 ν^2 δ ϵ U[c]) ** Muw[ν δ, n] +
     Expand@(-2 ν^4 b δ^3 ϵ U[u] + 2 ν^3 δ^2 ϵ UU[c, u]) ** Muw[ν δ, n] ** U[w] -
     (b ν^5 δ^4 ϵ)/2 U[u, u] ** Muw[ν δ, n] ** U[w, w]
   ] // ToDegree[
    n]]
```

0

```
Coefficient[t1, U[u, u, w, w]] /. ℏ → 1
```

0

```
Series[b δ^4 ∊ / (2 (1 + b δ)^5), {b, 0, 7}] // Normal
```

$$\frac{1}{2} b\,\delta^4 \epsilon - \frac{5}{2} b^2 \delta^5 \epsilon + \frac{15}{2} b^3 \delta^6 \epsilon - \frac{35}{2} b^4 \delta^7 \epsilon + 35\, b^5 \delta^8 \epsilon - 63\, b^6 \delta^9 \epsilon + 105\, b^7 \delta^{10} \epsilon$$

```
Simplify[(1/(1 + b δ) + (-b δ^2 ∊)/(1 + b δ)^3)]
```

$$\frac{1}{1+b\,\delta} - \frac{b\,\delta^2 \epsilon}{(1+b\,\delta)^3}$$

## Hard core uw relations.

```
With[{n = 3}, Simp[
   UExp[α ℏ U@w, n] ** UExp[β U@u, n] - ⅇ^(-b ℏ α β) UExp[β U@u, n] ** UExp[α ℏ U@w, n]
  ] // ToDegree[n]]
```

$$2\,\alpha\,\beta\,\epsilon\,U[c] + \alpha\,\beta^2 \epsilon\,U[u] + \alpha^2 \beta\,\epsilon\,U[w] + 2\,\alpha^2 \beta\,\epsilon\,U[c, w] + 2\,\alpha\,\beta^2 \epsilon\,U[u, c]$$

```
n = 14; With[{ν = (1 + b δ)^-1},
 LHS = Expand[ⅇ^(b ℏ ν α β) Sum[
     (α ℏ)^n1 δ^n2 β^n3 / (n1! n2! n3!) UU[w^(n1+n2), u^(n2+n3)],
     {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
    ]];
 RHS = Expand[ν Sum[
     (ν β)^n1 (ν δ)^n2 (ν α ℏ)^n3 / (n1! n2! n3!) UU[u^(n1+n2), w^(n2+n3)],
     {n1, 0, n}, {n2, 0, n - n1}, {n3, 0, n - n1 - n2}
    ]];
]

t1 = With[{ν = (1 + b δ)^-1}, Simp[ToDegree[14][-LHS +
      RHS - 1/2 b ∊ ν^2 (2 δ^2 + 4 α β δ ν + α^2 β^2 ν^2) RHS +
      2 ∊ ν (δ + α β ν) U[c] ** RHS - β (1 + 2 b δ) ∊ ν^3 (2 δ + α β ν) U[u] ** RHS +
      α ∊ ν^3 (2 δ + α β ν) RHS ** U[w] + Expand[2 β δ ν^2 ∊ UU[c, u]] ** RHS +
      2 α δ ν^2 ∊ U[c] ** RHS ** U[w] - 1/2 β^2 δ (2 + 3 b δ) ∊ ν^4 U[u, u] ** RHS +
      1/2 α^2 δ (2 + b δ) ∊ ν^4 RHS ** U[w, w] - 2 b δ^2 ∊ ν^3 (δ + α β ν) U[u] ** RHS ** U[w] +
      Expand[2 δ^2 ν^2 ∊ UU[c, u]] ** RHS ** U[w] - β δ^2 (1 + 2 b δ) ∊ ν^4 U[u, u] ** RHS ** U[w] +
      α δ^2 ν^4 ∊ U[u] ** RHS ** U[w, w] - 1/2 b δ^4 ν^4 ∊ U[u, u] ** RHS ** U[w, w]
    ]]];
t1 /. ∊ → 0
```

0

```
Expand[t1] /. a_.x_U /; (Count[x, u] + Exponent[a, b] + Exponent[a, α]) > 14 :> 0
```

$\frac{1365}{2} b^{12} \alpha^2 \beta^2 \delta^{11} \epsilon\, U[\,] - 910 b^{13} \alpha \beta \delta^{13} \epsilon\, U[\,] - 210 b^{13} \alpha \beta \delta^{13} \epsilon\, U[c] - \frac{3003}{2} b^{11} \alpha^2 \beta^3 \delta^{10} \epsilon\, U[u] + \cdots 2040 \cdots + \frac{\cdots 1 \cdots}{159667200} + \frac{\alpha^2 \cdots 3 \cdots}{239500800} + \frac{\alpha \cdots 4 \cdots}{222393600} + \frac{1}{6227020800} 41 \alpha \delta^{14} \epsilon\, U[u, u, u, u, u, u, u, u, u, u, u, u, u, w, w, w, w, w, w, w, w, w, w, w, w, w, w] + \frac{1}{3113510400} \alpha \delta^{14} \epsilon\, U[u, u, u, u, u, u, u, u, u, u, u, u, c, w, w, w, w, w, w, w, w, w, w, w, w, w, w]$

large output | show less | show more | show all | set size limit...

```
Take[Cases[t1, _U, ∞] // Sort, 30]
```

{U[], U[c], U[u], U[w], U[c, w], U[u, c], U[u, u], U[u, w], U[w, w], U[c, w, w], U[u, c, w], U[u, u, c], U[u, u, u], U[u, u, w], U[u, w, w], U[w, w, w], U[c, w, w, w], U[u, c, w, w], U[u, u, c, w], U[u, u, u, c], U[u, u, u, u], U[u, u, u, w], U[u, u, w, w], U[u, w, w, w], U[w, w, w, w], U[c, w, w, w, w], U[u, c, w, w, w], U[u, u, c, w, w], U[u, u, u, c, w], U[u, u, u, u, c]}

```
t2 = Collect[Coefficient[t1, U[w, w, w, w]] // Simplify, β, Simplify]
```

$-\frac{2145}{16} b^8 \alpha^6 \beta^2 \delta^7 (-1 + 2 b \delta) \epsilon + \frac{143}{12} b^9 \alpha^5 \beta \delta^9 (-205 + 272 b \delta) \epsilon - \frac{91}{12} b^{10} \alpha^4 \delta^{11} (-715 + 552 b \delta) \epsilon$

```
Sum[FindSequenceFunction[CoefficientList[
      1 - 3 b δ + 5 b^2 δ^2 - 5 b^3 δ^3 + 14 b^5 δ^5 - 42 b^6 δ^6 + 90 b^7 δ^7 - 165 b^8 δ^8 + 275 b^9 δ^9 - 429 b^10 δ^10
      , b] /. δ → 1][k + 1] (b δ)^k, {k, 0, ∞}]
```

$\frac{1 + 2 b \delta}{(1 + b \delta)^5}$

```
t3 = With[{ν = (1 + b δ)^-1}, Series[
      1/2 b δ^4 ν^5 ϵ,
      {b, 0, 12}]] // Normal // Simplify
```

$\frac{1}{2} b \delta^4 (1 - 5 b \delta + 15 b^2 \delta^2 - 35 b^3 \delta^3 + 70 b^4 \delta^4 - 126 b^5 \delta^5 + 210 b^6 \delta^6 - 330 b^7 \delta^7 + 495 b^8 \delta^8 - 715 b^9 \delta^9 + 1001 b^{10} \delta^{10} - 1365 b^{11} \delta^{11}) \epsilon$

```
Series[t2 - t3, {b, 0, 8}]
```

$-\frac{1}{2} (\delta^4 \epsilon) b + \frac{5}{2} \delta^5 \epsilon b^2 - \frac{15}{2} (\delta^6 \epsilon) b^3 + \frac{35}{2} \delta^7 \epsilon b^4 - 35 (\delta^8 \epsilon) b^5 + 63 \delta^9 \epsilon b^6 - 105 (\delta^{10} \epsilon) b^7 + \left(\frac{2145}{16} \alpha^6 \beta^2 \delta^7 \epsilon + 165 \delta^{11} \epsilon\right) b^8 + O[b]^9$

```
Simplify[β δ^2 (1 + 2 b δ) ν^4 ϵ]
```

$\beta \delta^2 (1 + 2 b \delta) \epsilon \nu^4$

---

```
Λ = With[{vs = {1, u0, w0, u0^2, w0^2, u0^2 w0, u0 w0^2, u0^2 w0^2, c0, c0 u0, c0 w0, c0 u0 w0}},
   Table[a_i, {i, Length@vs}].vs
  ]
```

$a_1 + a_9 c_0 + a_2 u_0 + a_{10} c_0 u_0 + a_4 u_0^2 + a_3 w_0 + a_{11} c_0 w_0 + a_{12} c_0 u_0 w_0 + a_6 u_0^2 w_0 + a_5 w_0^2 + a_7 u_0 w_0^2 + a_8 u_0^2 w_0^2$

$$\textbf{With}\left[\{n = 5,\ \mu = 1 + b_1\,\delta\},\ \textbf{ToDegree}[n]\left[\right.\right.$$
$$\mathbb{O}\left[n,\ e^{\hbar\,\alpha\,w_0+\beta\,u_0+\delta\,u_0\,w_0},\ \{w_0,\ u_0\}\to 1\right] -$$
$$\mathbb{O}\left[n,\ \mu^{-1}\left(1+\epsilon\,\mu^{-1}\,\Lambda\right)e^{(-b_1\,\hbar\,\alpha\,\beta+\hbar\,\alpha\,w_0+\beta\,u_0+\delta\,u_0\,w_0)/\mu},\ \{u_0,\ c_0,\ w_0\}\to 1\right]$$
$$\left.\left.\right]\right]\ //$$
$$\textbf{Simp}$$

$$\left(-\epsilon\,\hbar^2\,a_1-\delta^2\,\epsilon\,\hbar^4\,b_1-2\,\alpha\,\beta\,\delta\,\epsilon\,\hbar^5\,b_1+2\,\delta\,\epsilon\,\hbar^4\,a_1\,b_1+\alpha\,\beta\,\epsilon\,\hbar^5\,a_1\,b_1\right)U[\,]+$$
$$\left(2\,\delta\,\epsilon\,\hbar^2+2\,\alpha\,\beta\,\epsilon\,\hbar^3-\epsilon\,\hbar^2\,a_9-4\,\delta^2\,\epsilon\,\hbar^4\,b_1-8\,\alpha\,\beta\,\delta\,\epsilon\,\hbar^5\,b_1+2\,\delta\,\epsilon\,\hbar^4\,a_9\,b_1+\alpha\,\beta\,\epsilon\,\hbar^5\,a_9\,b_1\right)U[c_1]+$$
$$\left(2\,\beta\,\delta\,\epsilon\,\hbar^4+\alpha\,\beta^2\,\epsilon\,\hbar^5-\beta\,\epsilon\,\hbar^4\,a_1-\epsilon\,\hbar^4\,a_2\right)U[u_1]+$$
$$\left(2\,\alpha\,\delta\,\epsilon\,\hbar^3+\alpha^2\,\beta\,\epsilon\,\hbar^4-\alpha\,\epsilon\,\hbar^3\,a_1-\epsilon\,\hbar^2\,a_3-9\,\alpha\,\delta^2\,\epsilon\,\hbar^5\,b_1+3\,\alpha\,\delta\,\epsilon\,\hbar^5\,a_1\,b_1+2\,\delta\,\epsilon\,\hbar^4\,a_3\,b_1+\alpha\,\beta\,\epsilon\,\hbar^5\,a_3\,b_1\right)$$
$$U[w_1]+\left(4\,\alpha\,\delta\,\epsilon\,\hbar^3+2\,\alpha^2\,\beta\,\epsilon\,\hbar^4-\alpha\,\epsilon\,\hbar^3\,a_9-\epsilon\,\hbar^2\,a_{11}-12\,\alpha\,\delta^2\,\epsilon\,\hbar^5\,b_1+3\,\alpha\,\delta\,\epsilon\,\hbar^5\,a_9\,b_1+2\,\delta\,\epsilon\,\hbar^4\,a_{11}\,b_1+\right.$$
$$\left.\alpha\,\beta\,\epsilon\,\hbar^5\,a_{11}\,b_1\right)U[c_1,\ w_1]+\left(4\,\beta\,\delta\,\epsilon\,\hbar^4+2\,\alpha\,\beta^2\,\epsilon\,\hbar^5-\beta\,\epsilon\,\hbar^4\,a_9-\epsilon\,\hbar^4\,a_{10}\right)U[u_1,\ c_1]+$$
$$\left(4\,\delta^2\,\epsilon\,\hbar^4+8\,\alpha\,\beta\,\delta\,\epsilon\,\hbar^5-\delta\,\epsilon\,\hbar^4\,a_1-\alpha\,\beta\,\epsilon\,\hbar^5\,a_1-\alpha\,\epsilon\,\hbar^5\,a_2-\beta\,\epsilon\,\hbar^4\,a_3\right)U[u_1,\ w_1]+$$
$$\left(3\,\alpha^2\,\delta\,\epsilon\,\hbar^4+\alpha^3\,\beta\,\epsilon\,\hbar^5-\frac{1}{2}\,\alpha^2\,\epsilon\,\hbar^4\,a_1-\alpha\,\epsilon\,\hbar^3\,a_3-\epsilon\,\hbar^2\,a_5+3\,\alpha\,\delta\,\epsilon\,\hbar^5\,a_3\,b_1+2\,\delta\,\epsilon\,\hbar^4\,a_5\,b_1+\alpha\,\beta\,\epsilon\,\hbar^5\,a_5\,b_1\right)$$
$$U[w_1,\ w_1]+\left(3\,\alpha^2\,\delta\,\epsilon\,\hbar^4+\alpha^3\,\beta\,\epsilon\,\hbar^5-\frac{1}{2}\,\alpha^2\,\epsilon\,\hbar^4\,a_9-\alpha\,\epsilon\,\hbar^3\,a_{11}+3\,\alpha\,\delta\,\epsilon\,\hbar^5\,a_{11}\,b_1\right)U[c_1,\ w_1,\ w_1]+$$
$$\left(4\,\delta^2\,\epsilon\,\hbar^4+8\,\alpha\,\beta\,\delta\,\epsilon\,\hbar^5-\delta\,\epsilon\,\hbar^4\,a_9-\alpha\,\beta\,\epsilon\,\hbar^5\,a_9-\alpha\,\epsilon\,\hbar^5\,a_{10}-\beta\,\epsilon\,\hbar^4\,a_{11}-\epsilon\,\hbar^4\,a_{12}\right)U[u_1,\ c_1,\ w_1]+$$
$$\left(9\,\alpha\,\delta^2\,\epsilon\,\hbar^5-\alpha\,\delta\,\epsilon\,\hbar^5\,a_1-\delta\,\epsilon\,\hbar^4\,a_3-\alpha\,\beta\,\epsilon\,\hbar^5\,a_3-\beta\,\epsilon\,\hbar^4\,a_5-\epsilon\,\hbar^4\,a_7\right)U[u_1,\ w_1,\ w_1]+$$
$$\left(2\,\alpha^3\,\delta\,\epsilon\,\hbar^5-\frac{1}{6}\,\alpha^3\,\epsilon\,\hbar^5\,a_1-\frac{1}{2}\,\alpha^2\,\epsilon\,\hbar^4\,a_3-\alpha\,\epsilon\,\hbar^3\,a_5+3\,\alpha\,\delta\,\epsilon\,\hbar^5\,a_5\,b_1\right)U[w_1,\ w_1,\ w_1]+$$
$$\left(\frac{4}{3}\,\alpha^3\,\delta\,\epsilon\,\hbar^5-\frac{1}{6}\,\alpha^3\,\epsilon\,\hbar^5\,a_9-\frac{1}{2}\,\alpha^2\,\epsilon\,\hbar^4\,a_{11}\right)U[c_1,\ w_1,\ w_1,\ w_1]+$$
$$\left(6\,\alpha\,\delta^2\,\epsilon\,\hbar^5-\alpha\,\delta\,\epsilon\,\hbar^5\,a_9-\delta\,\epsilon\,\hbar^4\,a_{11}-\alpha\,\beta\,\epsilon\,\hbar^5\,a_{11}-\alpha\,\epsilon\,\hbar^5\,a_{12}\right)U[u_1,\ c_1,\ w_1,\ w_1]+$$
$$\left(-\alpha\,\delta\,\epsilon\,\hbar^5\,a_3-\delta\,\epsilon\,\hbar^4\,a_5-\alpha\,\beta\,\epsilon\,\hbar^5\,a_5-\alpha\,\epsilon\,\hbar^5\,a_7\right)U[u_1,\ w_1,\ w_1,\ w_1]+$$
$$\left(-\frac{1}{6}\,\alpha^3\,\epsilon\,\hbar^5\,a_3-\frac{1}{2}\,\alpha^2\,\epsilon\,\hbar^4\,a_5\right)U[w_1,\ w_1,\ w_1,\ w_1]-\frac{1}{6}\,\alpha^3\,\epsilon\,\hbar^5\,a_{11}\,U[c_1,\ w_1,\ w_1,\ w_1,\ w_1]-$$
$$\alpha\,\delta\,\epsilon\,\hbar^5\,a_{11}\,U[u_1,\ c_1,\ w_1,\ w_1,\ w_1]-\alpha\,\delta\,\epsilon\,\hbar^5\,a_5\,U[u_1,\ w_1,\ w_1,\ w_1,\ w_1]-\frac{1}{6}\,\alpha^3\,\epsilon\,\hbar^5\,a_5\,U[w_1,\ w_1,\ w_1,\ w_1,\ w_1]$$