Pensieve header: NOE-1 using vcw conventions

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2016-11"];
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

NOE-1 Programs

```
Λ[k_] := -(1 + t_k)/4 (4 v_k c_k w_k δ^2 μ^2 + δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) + v_k^2 w_k^2 δ^3 (1 + 3 μ) + (1 - t_k) (2 (α β + δ μ)^2 - α^2 β^2) +
    2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) + 4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ));
```

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α,∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

```
CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
    𝔼[ω /. b_l_ ⧴ Log[t_l], L, Q /. b_l_ ⧴ Log[t_l], P /. b_l_ ⧴ Log[t_l]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] := CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] ≡ 𝔼[ω2_, L2_, Q2_, P2_] := (ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2);
```

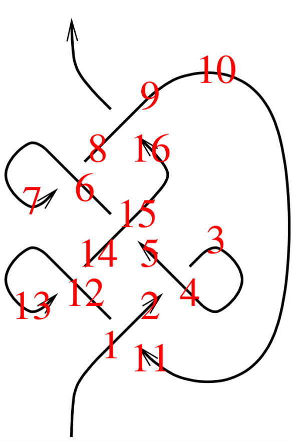
```
N_{c_j_ (x:v|w)_i_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ⅇ^γ β x_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω ⅇ^γ β x_k + (Q /. x_i → 0), ⅇ^-q DP_{c_j→D_γ, x_i→D_β}[P][ⅇ^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{x_i} Q}]];
```

```
N_{w_i_ v_j_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ((1 - t_k) α β + β v_k + δ v_k w_k + α w_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0), μ^4 ⅇ^-q DP_{w_i→D_α, v_j→D_β}[P][ⅇ^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
     {α → ω^-1 (∂_{w_i} Q /. v_j → 0), β → ω^-1 (∂_{v_j} Q /. w_i → 0), δ → ω^-1 ∂_{w_i,v_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, z},
  Z // N_{w_i v_j→x} // N_{c_i v_x→x} // N_{w_x c_j→x} // ReplaceAll[z_{i|j|x} → z_k] // CF]
```

```
R^+_{i_,j_} := 𝔼[1, b_i c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2 / 4];
R^-_{i_,j_} := 𝔼[1, -b_i c_j, -t_i^-1 v_i w_j, -c_i c_j + t_i^-1 v_i c_j w_j - t_i^-2 v_i^2 w_j^2 / 4];
ur_i_ := 𝔼[t_i^(-1/2), 0, 0, c_i t_i^-2]; nr_i_ := 𝔼[t_i^(1/2), 0, 0, -c_i t_i^2];
ul_ = nl_ = 𝔼[1, 0, 0, 0];
```

The Trefoil



$$z2 = R^+_{1,11}\, R^-_{4,2}\, nr_3\, R^+_{15,5}\, R^-_{6,8}\, ur_7\, R^+_{9,16}\, nr_{10}\, R^-_{12,14}\, ur_{13};$$

$$\text{Do}[z2 = z2\ //\ m_{1,k\to 1},\ \{k, 2, 16\}];$$

$$z2 = z2\ /.\ a\_{}_1 \mapsto a$$

$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2c}{t^4}-\frac{1}{t^3}-\frac{6c}{t^3}+\frac{4}{t^2}+\frac{10c}{t^2}-\frac{10}{t}-\frac{8c}{t}-18t+8ct+14t^2-10ct^2-7t^3+6ct^3+2t^4-2ct^4+2vw-\frac{2vw}{t^4}+\frac{4vw}{t^3}-\frac{6vw}{t^2}+\frac{2vw}{t}-6tvw+4t^2vw-2t^3vw\Big]$$

Meta-Associativity

$$Q0 = \mathbb{E}[1,\ \text{Sum}[l_{i,j}\, b_i\, c_j,\ \{i, 3\},\ \{j, 3\}],\ \text{Sum}[a_{i,j}\, v_i\, w_j,\ \{i, 3\},\ \{j, 3\}],\ 0]$$

$$t1 = Q0\ //\ m_{1,2\to 4}\ //\ m_{4,3\to 5}$$

$$t2 = Q0\ //\ m_{2,3\to 4}\ //\ m_{1,4\to 5}$$

$$t3 = (t1 \equiv t2)$$

$$\mathbb{E}[1,\ b_1 c_1 l_{1,1}+b_1 c_2 l_{1,2}+b_1 c_3 l_{1,3}+b_2 c_1 l_{2,1}+b_2 c_2 l_{2,2}+b_2 c_3 l_{2,3}+b_3 c_1 l_{3,1}+b_3 c_2 l_{3,2}+b_3 c_3 l_{3,3},\ v_1 w_1 a_{1,1}+v_1 w_2 a_{1,2}+v_1 w_3 a_{1,3}+v_2 w_1 a_{2,1}+v_2 w_2 a_{2,2}+v_2 w_3 a_{2,3}+v_3 w_1 a_{3,1}+v_3 w_2 a_{3,2}+v_3 w_3 a_{3,3},\ 0]$$

$$\mathbb{E}\big[1-a_{2,1}+t_5 a_{2,1}-t_5^{l_{1,2}+l_{2,2}+l_{3,2}} a_{3,1}+t_5^{1+l_{1,2}+l_{2,2}+l_{3,2}} a_{3,1}-a_{2,2}a_{3,1}+2t_5 a_{2,2}a_{3,1}-t_5^2 a_{2,2}a_{3,1}-a_{3,2}+t_5 a_{3,2}+a_{2,1}a_{3,2}-2t_5 a_{2,1}a_{3,2}+t_5^2 a_{2,1}a_{3,2},\ b_5 c_5 l_{1,1}+\langle\langle 7\rangle\rangle+b_5 c_5 l_{3,3},\ \langle\langle 1\rangle\rangle,\ \langle\langle 17315\rangle\rangle+t_5^{\langle\langle 1\rangle\rangle} v_5^2 w_5^2 a_{2,1}^2 a_{2,2}^2 a_{3,2}^2 a_{3,3}^2-\frac{1}{4} t_5^{6+\langle\langle 2\rangle\rangle+2 l_{3\langle\langle 1\rangle\rangle}\langle\langle 1\rangle\rangle} v_5^2 w_5^2 a_{2,1}^2 a_{2,2}^2 a_{3,2}^2 a_{3,3}^2\big]$$

large output | show less | show more | show all | set size limit...

$$\mathbb{E}\big[1-a_{2,1}+t_5 a_{2,1}-t_5^{l_{1,2}+l_{2,2}+l_{3,2}} a_{3,1}+t_5^{1+l_{1,2}+l_{2,2}+l_{3,2}} a_{3,1}-a_{2,2}a_{3,1}+2t_5 a_{2,2}a_{3,1}-t_5^2 a_{2,2}a_{3,1}-a_{3,2}+t_5 a_{3,2}+a_{2,1}a_{3,2}-2t_5 a_{2,1}a_{3,2}+t_5^2 a_{2,1}a_{3,2},\ b_5 c_5 l_{1,1}+\langle\langle 7\rangle\rangle+b_5 c_5 l_{3,3},\ \langle\langle 1\rangle\rangle,\ \langle\langle 17315\rangle\rangle+t_5^{\langle\langle 1\rangle\rangle} v_5^2 w_5^2 a_{2,1}^2 a_{2,2}^2 a_{3,2}^2 a_{3,3}^2-\frac{1}{4} t_5^{6+\langle\langle 2\rangle\rangle+2 l_{3\langle\langle 1\rangle\rangle}\langle\langle 1\rangle\rangle} v_5^2 w_5^2 a_{2,1}^2 a_{2,2}^2 a_{3,2}^2 a_{3,3}^2\big]$$

large output | show less | show more | show all | set size limit...

True

Computing Z_I

```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings xs and
    a length 2n list of rotation numbers rots. Crossing sites are indexed 1 through
    2n, and rots⟦k⟧ is the rotation between site k-1 and site k. RVK is also a casting
    operator converting to the RVK presentation from other knot presentations.";
RVK[pd_PD] := Module[{n, xs, x, rots, front, k},
   n = Length[pd];
   xs = List @@ pd /. x_X :> If[PositiveQ[x], Xp[x⟦4⟧, x⟦1⟧], Xm[x⟦2⟧, x⟦1⟧]];
   rots = Table[0, {2 n}];
   front = {0};
   For[k = 0, k < 2 n, ++k,
    If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → Catch[xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :> Throw[{l, k + 1, 1 - l}],
            Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots⟦l⟧; Throw[{1 - l, k + 1, l}])
           }]],
     If[MatchQ[front, {___, k, ___, -k, ___}], --rots⟦k + 1⟧]
    ]
   ];
   RVK[xs, rots]
  ];
RVK[K_] := RVK[PD[K]];
```

```
rot[_, 0] = 𝔼[1, 0, 0, 0];
rot[i_, 1] := ur_i;
rot[i_, n_Integer] /; n > 1 := Module[{y}, rot[i, n - 1] rot[y, 1] // m_{i,y→i}];
rot[i_, -1] := nr_i;
rot[i_, n_Integer] /; n < -1 := Module[{y}, rot[i, n + 1] rot[y, -1] // m_{i,y→i}];
```

```
{rot[i, 2], rot[i, -2]}
```

$$\left\{\mathbb{E}\left[\frac{1}{t_i}, 0, 0, \frac{2 c_i}{t_i^4}\right], \mathbb{E}\left[t_i, 0, 0, -2 c_i t_i^4\right]\right\}$$

```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := Z[rvk] = Module[{todo, n, rots, ζ, done, st, x, ζ1, i, j, k, k1, k2, k3},
   {todo, rots} = List @@ rvk;
   AppendTo[rots, 0];
   n = Length[todo];
   ζ = 𝔼[1, 0, 0, 0];
   done = {0};
   st = Range[0, 2 n + 1];
   While[todo =!= {},
    {x} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
    Z$todo = todo; Z$x = x;
    {i, j} = List @@ x;
    ζ1 = Switch[Head[x],
       Xp, m_{j,k→j}[R^+_{i,j} (R^-_{k3,k} nr_{k1} ul_{k2} // m_{k,k1→k} // m_{k,k2→k} // m_{k,k3→k})],
       Xm, m_{j,k→j}[R^-_{i,j} (R^+_{k,k3} nr_{k1} ul_{k2} // m_{k,k1→k} // m_{k,k2→k} // m_{k,k3→k})]
      ];
    ζ1 = rot[k, rots[[i]]] ζ1 // m_{k,i→i}; rots[[i]] = 0;
    ζ1 = ζ1 rot[k, rots[[i + 1]]] // m_{i,k→i}; rots[[i + 1]] = 0;
    ζ1 = rot[k, rots[[j]]] ζ1 // m_{k,j→j}; rots[[j]] = 0;
    ζ1 = ζ1 rot[k, rots[[j + 1]]] // m_{j,k→j}; rots[[j + 1]] = 0;
    ζ *= ζ1;
    If[MemberQ[done, i], ζ = ζ // m_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
    If[MemberQ[done, i - 1], ζ = ζ // m_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
    If[MemberQ[done, j], ζ = ζ // m_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
    If[MemberQ[done, j - 1], ζ = ζ // m_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
    done = done ⋃ {i - 1, i, j - 1, j};
    todo = DeleteCases[todo, x]
   ];
   ζ /. {v_0 → v, c_0 → c, w_0 → w}
  ]
```

```
Z[Knot[3, 1]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

$$\mathbb{E}\Big[-1+\frac{1}{t_0}+t_0,\ 0,\ 0,\ -16+2\,v\,w-\frac{2}{t_0^4}+\frac{2\,c}{t_0^4}-\frac{2\,v\,w}{t_0^4}+\frac{7}{t_0^3}-\frac{6\,c}{t_0^3}+\frac{4\,v\,w}{t_0^3}-\frac{14}{t_0^2}+\frac{10\,c}{t_0^2}-\frac{6\,v\,w}{t_0^2}+\frac{18}{t_0}-\frac{8\,c}{t_0}+\frac{2\,v\,w}{t_0}+10\,t_0+8\,c\,t_0-6\,v\,w\,t_0-4\,t_0^2-10\,c\,t_0^2+4\,v\,w\,t_0^2+t_0^3+6\,c\,t_0^3-2\,v\,w\,t_0^3-2\,c\,t_0^4\Big]$$

```
t_ = t; Z[Knot["K11n34"] // Mirror]
```

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

$$\mathbb{E}\Big[1,\ 0,\ 0,\ -\frac{2}{t^3}+\frac{4}{t^2}-\frac{2}{t}-2\,t+4\,t^2-2\,t^3\Big]$$

```
t_ = t; Z[Knot["K11n42"] // Mirror]
```

$$\mathbb{E}\Big[1,\ 0,\ 0,\ -\frac{2}{t^3}+\frac{4}{t^2}-\frac{2}{t}-2\,t+4\,t^2-2\,t^3\Big]$$

```
Dynamic[$now]
```

```
t_ = t; tab7 = Table[$now = (K → Timing@Z[K]), {K, AllKnots[{3, 10}]}];
```

Extracting ρ_1

```
za[K_Knot] := Z[K][[1]];
zp[K_Knot] := Z[K][[4]];
```

```
Union@Table[za[K] == Alexander[K][t], {K, AllKnots[{3, 7}]}]
```

{True}

```
MatrixForm@
 Table[{za[K], Factor@(Coefficient[zp[K], c] / (-2 t za[K]^3 D[za[K], t])), Factor@(Coefficient[zp[K], v w] / (2 t/(1-t) za[K]^3 D[za[K], t]))}, {K, AllKnots[{3, 7}]}]
```

$$\begin{pmatrix}
-1+\frac{1}{t}+t & 1 & 1 \\
3-\frac{1}{t}-t & 1 & 1 \\
1+\frac{1}{t^2}-\frac{1}{t}-t+t^2 & 1 & 1 \\
-3+\frac{2}{t}+2t & 1 & 1 \\
5-\frac{2}{t}-2t & 1 & 1 \\
-3-\frac{1}{t^2}+\frac{3}{t}+3t-t^2 & 1 & 1 \\
5+\frac{1}{t^2}-\frac{3}{t}-3t+t^2 & 1 & 1 \\
-1+\frac{1}{t^3}-\frac{1}{t^2}+\frac{1}{t}+t-t^2+t^3 & 1 & 1 \\
-5+\frac{3}{t}+3t & 1 & 1 \\
3+\frac{2}{t^2}-\frac{3}{t}-3t+2t^2 & 1 & 1 \\
-7+\frac{4}{t}+4t & 1 & 1 \\
5+\frac{2}{t^2}-\frac{4}{t}-4t+2t^2 & 1 & 1 \\
-7-\frac{1}{t^2}+\frac{5}{t}+5t-t^2 & 1 & 1 \\
9+\frac{1}{t^2}-\frac{5}{t}-5t+t^2 & 1 & 1
\end{pmatrix}$$

```
etab = Get["../Talks/UNC-1610/etab.m"];
e[K_] := K /. etab;
```

```
MatrixForm@(mat = Table[{(*K[[1]]_K[[2]],za[K],*) Expand@Together@((zp[K] /. v | c | w → 0) / za[K]^2),
    Expand@(za[K] D[za[K], t]), e[K]}, {K, AllKnots[{3, 7}]}])
```

$$\begin{pmatrix}
-2-\frac{2}{t^2}+\frac{3}{t}+t & -1-\frac{1}{t^3}+\frac{1}{t^2}+t & \\
-\frac{1}{t^2}+\frac{3}{t}-3t+t^2 & -3-\frac{1}{t^3}+\frac{3}{t^2}+t & \\
-6-\frac{4}{t^4}+\frac{7}{t^3}-\frac{8}{t^2}+\frac{8}{t}+4t-2t^2+t^3 & -2-\frac{2}{t^5}+\frac{3}{t^4}-\frac{3}{t^3}+\frac{2}{t^2}+3t-3t^2+2t^3 & \\
-18-\frac{9}{t^2}+\frac{20}{t}+8t-t^2 & -6-\frac{4}{t^3}+\frac{6}{t^2}+4t & \\
-10-\frac{5}{t^2}+\frac{16}{t}-4t+3t^2 & -10-\frac{4}{t^3}+\frac{10}{t^2}+4t & \\
-16-\frac{3}{t^4}+\frac{15}{t^3}-\frac{28}{t^2}+\frac{28}{t}+4t+2t^2-3t^3+t^4 & -12-\frac{2}{t^5}+\frac{9}{t^4}-\frac{15}{t^3}+\frac{12}{t^2}+15t-9t^2+2t^3 & - \\
-\frac{2}{t^4}+\frac{9}{t^3}-\frac{19}{t^2}+\frac{18}{t}-18t+19t^2-9t^3+2t^4 & -18-\frac{2}{t^5}+\frac{9}{t^4}-\frac{19}{t^3}+\frac{18}{t^2}+19t-9t^2+2t^3 & \\
-12-\frac{6}{t^6}+\frac{11}{t^5}-\frac{14}{t^4}+\frac{16}{t^3}-\frac{16}{t^2}+\frac{15}{t}+9t-6t^2+4t^3-2t^4+t^5 & -3-\frac{3}{t^7}+\frac{5}{t^6}-\frac{6}{t^5}+\frac{6}{t^4}-\frac{5}{t^3}+\frac{3}{t^2}+5t-6t^2+6t^3-5t^4+3t^5 & \\
-60-\frac{23}{t^2}+\frac{59}{t}+29t-5t^2 & -15-\frac{9}{t^3}+\frac{15}{t^2}+9t & \\
56+\frac{1}{t^4}-\frac{8}{t^3}+\frac{20}{t^2}-\frac{37}{t}-67t+62t^2-44t^3+17t^4 & -15-\frac{8}{t^5}+\frac{18}{t^4}-\frac{21}{t^3}+\frac{15}{t^2}+21t-18t^2+8t^3 & 12 \\
112+\frac{8}{t^2}-\frac{52}{t}-108t+40t^2 & -28-\frac{16}{t^3}+\frac{28}{t^2}+16t & \\
-114-\frac{17}{t^4}+\frac{58}{t^3}-\frac{106}{t^2}+\frac{130}{t}+74t-34t^2+10t^3-t^4 & -28-\frac{8}{t^5}+\frac{24}{t^4}-\frac{36}{t^3}+\frac{28}{t^2}+36t-24t^2+8t^3 & -28 \\
-78-\frac{3}{t^4}+\frac{25}{t^3}-\frac{75}{t^2}+\frac{106}{t}+26t+3t^2-5t^3+t^4 & -40-\frac{2}{t^5}+\frac{15}{t^4}-\frac{39}{t^3}+\frac{40}{t^2}+39t-15t^2+2t^3 & -2 \\
22-\frac{2}{t^4}+\frac{15}{t^3}-\frac{40}{t^2}+\frac{36}{t}-64t+46t^2-15t^3+2t^4 & -50-\frac{2}{t^5}+\frac{15}{t^4}-\frac{43}{t^3}+\frac{50}{t^2}+43t-15t^2+2t^3 &
\end{pmatrix}$$

```
MatrixRank[mat]
```

2

```
NullSpace[mat]
```

$$\left\{\left\{\frac{t}{(-1+t)^2}, -\frac{t^2}{(-1+t)^2}, 1\right\}\right\}$$

```
ρ1[K_] := Expand@Together[-t (zp[K] /. v | c | w → 0)/((-1 + t)^2 za[K]^2) + t^2 za[K] D[za[K], t]/(-1 + t)^2]
```

$$\rho_1[K\_] := \text{Expand@Together}\left[-\frac{t\,(\text{zp}[K]\ /.\ v\mid c\mid w\to 0)}{(-1+t)^2\,\text{za}[K]^2}+\frac{t^2\,\text{za}[K]\,D[\text{za}[K],t]}{(-1+t)^2}\right]$$

```
Union@Table[Simplify[
   zp[K] == 1/((1 - t) t) za[K]^2 ((t - 1)^3 ρ1[K] + t^2 (1 - t - 2 c (1 - t) + 2 v w) za[K] D[za[K], t])
  ], {K, AllKnots[{3, 10}]}]
```

$$\text{zp}[K]==\frac{1}{(1-t)\,t}\text{za}[K]^2\left((t-1)^3\rho_1[K]+t^2\left(1-t-2c(1-t)+2vw\right)\text{za}[K]\,D[\text{za}[K],t]\right)$$

{True}

```
Union@Table[e[K] == ρ1[K], {K, AllKnots[{3, 10}]}]
```

{True}