# Cheat Sheet OneCo-1611



**(161121b)** Control alt-$\Delta$!

**(161121a)** Roland's 1-co middle $c$ data, from People/VanDerVeen/1CoMidCStraight10.nb. At $\{\omega, L, Q, P\} \to \mathbb{O}\left(\omega^{-1}e^{L+Q/\omega}(1+\epsilon\omega^{-4}P)\colon acw\right)$, verified at People/VanDerVeen/ NewVdVAlgebraAt17.nb:

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Λ = 4 a c w δ^2 μ^2 + δ (1 + μ) (w^2 α^2 + a^2 β^2) +
    a^2 w^2 δ^3 (1 + 3 μ) + (1 - t) (2 (α β + δ μ)^2 - α^2 β^2) +
    2 (α β + 2 δ μ + a w δ^2 (1 + 2 μ) + 2 c δ μ^2) (w α + a β) +
    4 (c μ^2 + a w δ (1 + μ)) (α β + δ μ);

Rp[i_, j_] := {1, b_i c_j, a_i w_j, a_i c_i w_j + c_i c_j + a_i^2 w_j^2 / 4}

Rm[i_, j_] :=
 {1, -b_i c_j, -t_i^-1 a_i w_j, -c_i c_j + t_i^-1 a_i c_j w_j - t_i^-2 a_i^2 w_j^2 / 4}

nr[i_] := {t_i^(1/2), 0, 0, -c_i t_i^2}
ur[i_] := {t_i^(-1/2), 0, 0, c_i t_i^-2}
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**(161112b)** Roland's $q\mathfrak{g}_0$ at `genus0co.nb`: $a = \zeta u$, $\Delta_{jk}(a) = a_j + t_j a_k$, $S(A) = -t^{-1}a$. Q. How does $\Delta_{q\mathfrak{g}_0}$ relates to $\Delta_{\mathfrak{g}_0}$? Is $KV(\mathfrak{g}_0)$ more than a projection of $KV(FL)$? Due to $e^{b_1c_2}$?

**(161112a)** With $\zeta = \frac{t-1}{b}$, an algebra isomorphism $q\mathfrak{g}_1 \to \mathfrak{g}_1$ via $a \mapsto \zeta u$, $\epsilon \mapsto -\frac{2\zeta}{t+1}\epsilon$? Roland on 161118: all simplicity advantages in his model arise from the $\zeta$-rescaling and the $cuw \to acw$ PBW change.

**(161110)** Roland's $q\mathfrak{g}_1$ at `NewVdVAlgebraAt16.nb`: $[c,u] = u$, $[c,w] = -w$, $[u,w] = (t-1) - \epsilon uw + 2\epsilon tc$, $\Delta_{jk}(b,c,u,w) = (b_j + b_k, c_j + c_k, t_k e^{\epsilon c_k}u_j + u_k, e^{\epsilon c_k}w_j + w_k)$, $S(b,c,u,w) = (-b, -c, -t^{-1}ue^{-\epsilon c}, -we^{-\epsilon c})$. Then $a := ue^{-\epsilon c}$ so $[c,a] = a$, $[a,w] = (t-1) + \epsilon(t+1)c$, $\Delta_{jk}(a) = t_k a_j + a_k e^{-\epsilon c_j}$, $S(a) = -t^{-1}(ae^{\epsilon c} + \epsilon a)$.

**(160920)** Roland's $s\mathfrak{g}_1$: $[w,c] = w$, $[u,c] = -u$, $[w,u] = e^{\epsilon c}$ ($s$ for September).

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1-Smidgen $sl_2$ Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with $b$ central and with $[w,c] = w$, $[c,u] = u$, and $[u,w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of $sl_2$: $\mathfrak{g}_1 \supset \langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset \langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset 0$. (note: $\deg(b,c,u,w,\epsilon) = (1,0,1,0,1)$)

0-Smidgen $sl_2$. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot] = 0, [c,u] = u, [c,w] = -w, [u,w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and $(b,u)$ is the dual basis of $(c,w)$.

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c,w\rangle/[w,c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c,w) \mapsto (0, c\wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b,u,c,w\rangle/\cdots$. Idea. Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w,c] = w$, $[w,b'] = -\epsilon w$, $[c,u] = u$, $[b',u] = -\epsilon u$, $[b',c] = 0$, and $[u,w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

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Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i}u_i w_j\right) \middle| i\colon u_i,\ j\colon c_j w_j\right)$.

The Big $\mathfrak{g}_0$ Lemma. With $[c,u] = u$, $[c,w] = -w$, $[u,w] = b$,

1. $\mathbb{O}(e^{\gamma c+\beta u}|uc) = \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u}|cu)$, meaning $e^{\beta u}e^{\gamma c} = e^{\gamma c}e^{e^{-\gamma}\beta u}$.
   Proof. $u \cdot \gamma c = \gamma \cdot (cu - u)\ldots$
2. $\mathbb{O}(e^{\gamma c+\alpha w}|wc) = \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w}|cw)$
3. $\mathbb{O}(e^{\alpha w+\beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$
4. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u} = e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu = (1+b\delta)^{-1}$
   (a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)
5. $\mathbb{O}(e^{\delta uw}|wu) = \mathbb{O}(\nu e^{\nu\delta uw}|uw)$
6. $\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu) = \mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

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The Big $\mathfrak{g}_1$ Lemma.

$$\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Here $\Lambda$ is for Λόγος, "a principle of order and knowledge", a balanced quartic in $\alpha$, $\beta$, $c$, $u$, and $w$:

$$\begin{aligned}\Lambda = &-b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta+2)\beta^2u^2/2\\ &- b\delta^4\nu^3u^2w^2/2 - \delta^2\nu^3(2b\delta+1)\beta u^2 w\\ &- \nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw\\ &+ \delta\nu^3(b\delta+2)\alpha^2w^2/2 + 2(\nu\alpha\beta+\delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw\\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha uw^2 + \nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

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**(160801a)** Roland's $sm_q sl(2)$ formulas, BBS:VanDerVeen-160731, with $q = e^\epsilon$, $t = e^b$: $b$ central, $[w,c] = w$, $[c,u] = u$, $wu - quw = 1 - te^{2\epsilon c}$ (at $\epsilon^2 = 0$: $[w,u] = \epsilon uw + 1 - t - 2\epsilon tc$), $R = \sum_{m,n}\frac{u^n(b+\epsilon c)^m \otimes c^m w^n}{m![n]_q!} \to \sum_{m,n}\frac{u^n(b+\epsilon c)^m\otimes c^m w^n}{m!n!}\left(1 - \frac{\epsilon}{2}\binom{n}{2}\right)$. Also, $\Delta(b,c,u,w) = (b_1+b_2, c_1+c_2, t_2e^{\epsilon c_2}u_1 + u_2, e^{\epsilon c_2}w_1 + w_2)$ and $S(b,c,u,w) = (-b,-c,-t^{-1}ue^{-\epsilon c}, -we^{-\epsilon c})$. Verified `VdVAlgebraAt1-Testing.nb`.

**(160801b)** $sm_0 sl(2)$ formulas, $t = e^b$: $b$ central, $[w,c] = w$, $[c,u] = u$, $wu - uw = 1 - t$, $R = \sum_{m,n}\frac{u^n b^m \otimes c^m w^n}{m!n!}$ (verified `VdVAlgebraAt0.nb`). Also, $\Delta(b,c,u,w) = (b_1+b_2, c_1+c_2, t_2u_1+u_2, w_1+w_2)$ and $S(b,c,u,w) = (-b,-c,-t^{-1}u,-w)$ (unverified).

**(160730)** Lessons from Roland: • There is an additional grading, with $ht(b,c,u,w) = (0,0,1,-1)$. • Rescale $u \to \frac{b}{e^b-1}u$. • A simple $R$-matrix for 1-co.

**(160628)** Figure out duality in $\mathfrak{g}_1$!

**(160622b)** $ad(-a_{12}) = \{c_1 \mapsto u_1w_2$, $c_2 \mapsto -u_1w_2$, $u_1 \mapsto \epsilon u_1c_2$, $u_2 \mapsto -(b_1-\epsilon c_1)u_2 + (b_2 - 2\epsilon c_2)u_1$, $w_1 \mapsto -b_1w_2 - \epsilon w_1c_2 + 2\epsilon c_1w_2$, $w_2 \mapsto (b_1 - \epsilon c_1)w_2\}$.

**Claim.** Over $\mathbb{Q}[\![\epsilon, b_i]\!]$ the following generate a sub-Lie algebra, sub-meta-monoid, and contains the $a_{ij}$'s:

$$\{1, c_i, u_i, w_i, u_iw_j\} \quad\text{and}$$

$$\epsilon\{c_ic_j, c_iu_j, c_iw_j, c_iu_jw_k, u_iu_jw_k, u_iw_jw_k, u_iu_jw_kw_l\}$$

**(160621)** 1-co low algebra ($\epsilon^2 = 0$): $a_{12} = I = b_1c_2 + u_1w_2 \in \mathfrak{b}_\epsilon^* \otimes \mathfrak{b}_\epsilon$

| | | | |
|---|---|---|---|
| $[w,c] = w$ | $[b,u] = -\epsilon u$ | $\delta c = 0$ | $\delta w = \epsilon(c\wedge w)$ |
| $[b,c] = 0$ | $[b,w] = \epsilon w$ | $[c,u] = u$ | $[u,w] = b - \epsilon c$ |

(verification in pensieve://2016-06)

Also, $ad(-a_{12}) = \{u_1 \mapsto \epsilon u_1c_2$, $u_2 \mapsto -b_1u_2 + b_2u_1 - \epsilon u_1c_2$, $b_1 \mapsto -\epsilon u_1w_2$, $b_2 \mapsto \epsilon u_1w_2$, $w_1 \mapsto -b_1w_2 - \epsilon w_1c_2 + \epsilon c_1w_2$, $w_2 \mapsto b_1w_2$, $c_1 \mapsto u_1w_2$, $c_2 \mapsto -u_1w_2\}$.

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**Recycling.**