Ricard Hain on Completions of Path Torsors of Moduli Spaces of Curves

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Lecture 1. Completions of groups & groupoids.

F : a field of char 0 : $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}_\ell$

An algebraic group over F is unipotent (unipt) if it is a closed subgroup of $\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ in $gl_N(F)$.

Equivalently: $U \subseteq gl(V)$ is unipt if V has a filtration

$$0 = V_0 \subseteq V_1 \ldots \subseteq V_r = V$$

by submodule s.t. V_j/V_{j-1} is trivial as a module.

Lie Alg: $\begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix} - I$ is nilpotent.

these are polynomial bijections.

So $U \cong \underline{u}$ as a variety

Lie alg of U

So $\mathcal{O}(U) \cong \mathrm{Sym}(\underline{u})$

Prounipotent groups: inverse limits of such:

$$U = \varprojlim U_\alpha \leftarrow \text{unipotent}$$

$$\underline{u} = \varprojlim \underline{u}_\alpha$$

$$\mathcal{O}(U) = \varinjlim \mathcal{O}(U_\alpha) = \varinjlim \mathrm{Sym}(u_\alpha)$$

<u>Example</u> V = f.d. v.s over F

$\{x_1, \dots x_n\}$ a basis

$T(V)$ = tensor alg $= FA(V)$

$= F\langle x_1, \dots x_n \rangle$

Aug $\epsilon: T(V) \longrightarrow F$ by $x_i \longrightarrow 0$

Aug. ideal $I = \ker \epsilon = \langle x_i \rangle$

$T(V)$ is a Hopf alg w/

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$$

$$\underset{\text{primitives}}{P} T(V) = \{u : \Delta u = 1 \otimes u + u \otimes 1\} = F\underset{\underset{\mathbb{L}(V)}{\shortparallel}}{L(V)}$$

I-adic topology $I \supset I^2 \supset \dots$

$T(V)^\wedge = \varprojlim T(V)/I^n$ = power series,

a complete Hopf algebra, so

$$\Delta : T(V)^\wedge \longrightarrow T(V)^\wedge \hat{\otimes} T(V)^\wedge$$

$$PT(V)^\wedge = \mathbb{L}(V)^\wedge$$

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gp-like elements:

$$GT(V)^\wedge := \{ g \in T(V)^\wedge : \epsilon(g) = 1,\ \Delta g = g \otimes g \}$$

$$\begin{array}{ccc} 1 + I^\wedge & \underset{\exp}{\overset{\log}{\rightleftarrows}} & I^\wedge \\ \cup\!\!/ & & \\ GT(V)^\wedge & \rightleftarrows & PT(V)^\wedge \end{array}$$

<u>Rem</u> $PT(V)^\wedge = \mathbb{L}(x_1 \ldots x_n)^\wedge = \varprojlim \mathbb{L}(x_i)/I^n \cap \mathbb{L}$

Unipotent completion: Γ : a discrete group

It's pro-unipotent completion $/\mathbb{F}$ $\Gamma^{un}_{/\mathbb{F}}$

is a pro unipotent gp $/\mathbb{F}$ w/ homomorphism

$$\Gamma \longrightarrow \Gamma^{un}(\mathbb{F})$$

w/ UMP (univ. mapping prop)

$$\Gamma \longrightarrow \Gamma^{un}_{/\mathbb{F}}$$

$$\forall \searrow \ \underset{\text{unip}/F}{\mathcal{U}} \ \overset{\exists !}{\dashleftarrow}$$

Group algebra $F\Gamma \xrightarrow{\epsilon} F \quad \epsilon(\gamma) = 1 \ \forall \gamma \in \Gamma$

Hopf w/ $\Delta(\gamma) = \gamma \otimes \gamma \qquad I = \ker \epsilon$

I-adic completion : $F\Gamma^\wedge = \varprojlim F\Gamma / I^n$

a complete Hopf algebra. [appendix A or B to paper of Quillen in the Annals ~1969]

Again exp & log make nice bijections between primitives & gp-like.

Quillen's Construction:

$$\Gamma^{un}_{/F} := G F\Gamma^\wedge$$

Example $\Gamma = \langle x_1, \ldots x_n \rangle$ = free group.

$\Theta : \Gamma \longrightarrow \mathbb{Q}\langle\langle X_1, \ldots\ldots X_n \rangle\rangle \qquad X_j$ primitives

via $x_j \longmapsto e^{X_j}$

By the univ. property,

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[dr] & G\mathbb{Q}\langle\langle X_1 \ldots X_n \rangle\rangle \\ & \Gamma^{un}_{/F} \arrow[u, dashed] \end{tikzcd}$$

an isomorphism!

So $\mathrm{Lie}\langle x_1 \ldots x_n\rangle^{un}_{/\mathbb{Q}} = \mathbb{L}(X_1 \ldots X_n)^\wedge$

Γ f.g. , $\Gamma = \langle x_1 \ldots x_n \mid r_1, \ldots, r_m\rangle$

$\Gamma^{un}_{/F}$ is a quotient of the unipotent completion of the free group.

$$\theta : \langle x_1 \ldots x_n\rangle \longrightarrow \mathbb{Q}\langle\langle X_1 \ldots X_n\rangle\rangle$$

$$r \longmapsto \theta(r) \longrightarrow \log\theta(r) \ldots$$

$$\mathrm{Lie}\,\Gamma^{un} = \mathbb{L}(X_1 \ldots X_n)^\wedge / (\log\theta(r_j))$$

Tannakian approach

G: a group , F : Field of char 0.
(appropriate)

$\mathrm{Rep}_F(G) =$ Category of f.d. reps of G/F.

Essential Features: 1. It's Abelian, F-linear, kers, cokers

2. tensor product (comm. assoc. naturally)

3. Has unit, the 1D trivial rep.

$$V \otimes 1 \cong 1 \otimes V \cong V.$$

4. Has duals. $V^*: \mathrm{Hom}_F(V \to 1)$

5. There is a faithful functor

$$\mathrm{Rep}_F(G) \longrightarrow \mathrm{Vect}_F \quad \left(\begin{array}{l}\text{respects } \otimes \\ \text{\& duals}\end{array}\right)$$

These $\underbrace{\text{are}}_{1-4}$ the axioms of an $\underbrace{}_{\text{neutral}}$ F-linear Tannakian category.

<u>Def</u> If T Tannakian then a fiber functor is a faithful functor

$$w: T \longrightarrow \mathrm{Vect}_F$$

that preserves $\otimes$ & duals.

<u>Basic</u> <u>Theorem</u> If T is a neutral Tannakian category w/ fiber functor w then T is equivalent to $\mathrm{Rep}_F(G)$, where $G = \mathrm{Aut}(\otimes_w)$.

<u>Notation</u> $G = \pi_1(T, w)$

Example Γ = discrete group

T = category of unipotent reps of Γ.

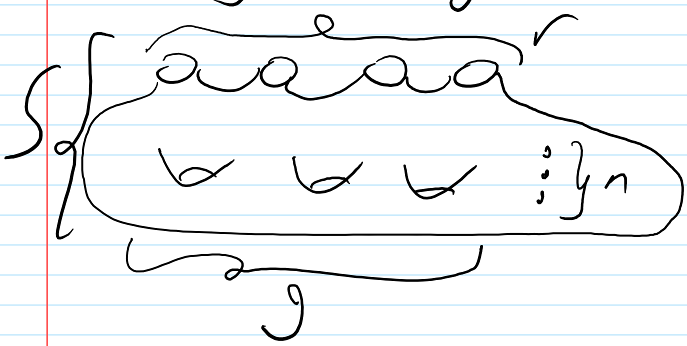
it is Tannakian !, $\pi_1(T) = \Gamma^{un}$

Problems w/ unipotent completions

(1) $H_1(\Gamma, F) = H_1(\Gamma^{un}_{/F})$ so $Sl_2(\mathbb{Z})^{un}_{/F} = 1$

In general, if H_1 is 1D, Γ^{un} is boring.

Mapping class groups



$P = \{x_1 \dots x_n\}$

$$\Gamma_{g,n+p} = \pi_0(\mathrm{Diff}^+(S, \partial S \cup P))$$

e.g. $\Gamma_{0,n+p} = P_n \qquad \Gamma_{1,1} = Sl_2(\mathbb{Z})$

Fact $\quad H_1(\Gamma_{g,n+\vec{r}}) = 0 \quad$ if $g \geq 1$

How 2

$$\Gamma_{g,n+\vec{r}} \longrightarrow \Gamma_g \longrightarrow \mathrm{Aut}(H_1(S,\mathbb{R}))$$
$$\cong$$
$$Sp_g(\mathbb{R}) \sim Sp(H)$$

Fact: $\quad \Gamma_{g,n+\vec{r}} \twoheadrightarrow Sp_g(\mathbb{Z})$

image is

$\downarrow$ Zariski dense

$$\begin{tikzcd} & \arrow[d, "\text{Zariski dense}"] \\ \text{image is Zariski dense} \arrow[r] & Sp_g(H) \end{tikzcd}$$

Relative unipotent completions

Γ : a discrete group, R = reductive group/F

$$\rho : \Gamma \longrightarrow R(F) \qquad \left(\text{example above}\right)$$

The completion of Γ rel. ρ:

1. an affine F-group $\mathcal{G}$ which is an extension

$$1 \longrightarrow \underset{\substack{\uparrow \\ \text{pro-unipotent}}}{U} \longrightarrow \mathcal{G} \longrightarrow R \longrightarrow 1$$

2. A homomorphism

$$\hat{\rho} : \Gamma \longrightarrow \mathcal{G}(F) \qquad \left[\begin{array}{c}\text{necessarily} \\ \text{Zariski} \\ \text{dense}\end{array}\right]$$

s.t. given

$$\begin{tikzcd} & U \arrow[r, dashed] \arrow[d, dashed] & \Gamma \arrow[drr] & & \\ 1 \arrow[r] & U \arrow[r] & G \arrow[rr] & & R \arrow[r] & 1 \end{tikzcd}$$

A Tannakian construction:

T : reps V of Γ w/ filtration

$$0 = V_0 \subseteq V_1 \quad \ldots \quad \subseteq V_r = V$$

s.t. Γ acts on V_j/V_{j-1} through R

$$V_j/V_{j-1} \circlearrowleft R \xleftarrow{\rho} \Gamma$$

$$\mathfrak{g} = \pi_1(T, w)$$

<u>Def</u> $\mathfrak{g}_{g,n+\vec{r}} :=$ Sp_g-completion of $\Gamma_{g,n+\vec{r}}$.

$g=0$: ordinary unipt completions.

$g=1$: Lecture 4.

$g=2$: know 1/2

$g\geq 3$: have explicit presentations.

Mixed Hodge Theory for non-specialists.

A $\mathbb{Q}$-Hodge structure V of weight m
is $\underbrace{\text{a}}_{\text{f.d.}}$ $\mathbb{Q}$ v.s., $\underbrace{\text{of}}_{V_{\mathbb{Q}}}$ and a filtration of $V_{\mathbb{C}}$

"The Hodge filtration"

$$V_{\mathbb{C}} = F^a V \supseteq F^{a+1} \supseteq \ldots \supseteq F^b V = 0$$

s.t.

$$V_{\mathbb{C}} = \bigoplus_{p+q=m} \underbrace{F^pV \cap \overline{F^qV}}_{V^{p,q},\ \overline{V^{p,q}} = V_{q,p}} \qquad V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C} \quad z \mapsto \bar{z}$$

$$F^pV = \bigoplus_{s\geq p} V^{s,m-s}$$

Example: X = smooth projective variety,

e.g., a $\underbrace{\text{Riemann surface}}_{\text{compact}}$

$V = H^n(X; \mathbb{Q})$

$V^{p,q} = H^{p,q}(X) = \dfrac{\text{closed } (p,q)\text{-forms}}{\text{exact } (p,q)\text{-forms}}$

Hodge's Thm

$$H^m(X;\mathbb{C}) = \bigoplus_{p+q} H^{p,q}(X)$$

Aside: $\mathbb{Q}(n)$ w/ $\begin{cases}\mathbb{Q} \\ V^{-n,-n}\end{cases}$ of wt $-2n$

Morphisms 1. preserve $\mathbb{Q}$ str.

2. preserve the Hodge filtration

Semi-simple mixed Hodge structures:

. . . .

Polarizations . . . .

The $\underbrace{\text{category of s.s. mixed H.S.}}_{MHS^{ss}}$ is Tannakian,

so it is $\mathrm{Rep}\,\pi_1(MHS^{ss}, w)$

Mixed Hodge structures:

f.d over $\mathbb{Q}$

$$V = (V_{\mathbb{Q}}, W_\bullet, F^\bullet)$$

increasing filtration, the wt filtration

decreasing "Hodge filtration"

$$V_{\mathbb{C}} = F^l \supseteq F^{l+1} \supseteq \dots \supseteq F^t = 0$$

s.t.

$$Gr_m^W V = \left(\frac{W_m V_{\mathbb{Q}}}{W_{m+1} V_{\mathbb{Q}}}, \text{ induced } F^\bullet \right)$$

is a Hodge str of wt m.

Thm (Deligne) X cmplx algebraic variety

$H^\bullet(X;\mathbb{Q})$ has a canon. MHS, graded polarizable.

MHS = category of graded polarizable MHSs.

It is neutral Tannakian $\quad MHS^{ss} \hookrightarrow MHS$

$$1 \to U \to \pi_1(MHS, w) \to \pi_1(MHS^{ss}) \to 1$$