The burau representation

```
burau_n_[0] := Table[0, {n}, {n}];
burau_n_[c_ * (α_a | α_b | α_NonCommutativeMultiply)] := c * burau_n[α];
burau_n_[expr_Plus] := burau_n /@ expr;
burau_n_[expr_NonCommutativeMultiply] := Dot @@ (burau_n /@ expr);
burau_n_[b[x_, y_]] := burau_n[x ** y - y ** x];
burau_n_[a[i_, j_]] := Normal@SparseArray[
    {{j, j} → b_i, {i, j} → -b_i}, {n, n}
   ];
deburau[m_] /; MatrixQ[m] ∧
    (Equal @@ Dimensions[m]) ∧ ({0} === Union[Simplify[Plus @@ m]]) := Sum[
   If[i == j, 0, -m[[i, j]] a[i, j] / b_i],
   {i, Length@m}, {j, Length@m}
  ];
```

```
burau_5[a[1, 3]] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
burau_5[a[1, 3]] // deburau
```

a[1, 3]

```
burau_5[a[1, 2] ** a[2, 3]] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & b_1 b_2 & 0 & 0 \\ 0 & 0 & -b_1 b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
burau_5[b[a[1, 2], a[1, 3]]]
```

{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}

```
burau_5[b[a[1, 3], a[2, 3]]]
```

{{0, 0, $-b_1 b_2$, 0, 0}, {0, 0, $b_1 b_2$, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}

```
burau_5[b[a[1, 3], a[2, 3]]] // deburau
```

-a[2, 3] b_1 + a[1, 3] b_2

```
burau_5[b[a[1, 2], a[2, 3]] + b[a[1, 3], a[2, 3]]]
```

{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}

The adjoint representation

```
ad_n_[x_][y_] := deburau[burau_n[x].burau_n[y] - burau_n[y].burau_n[x]];
```

```
{ad_5[a[1, 2]][a[3, 4]], ad_5[a[1, 3]][a[2, 3]], ad_5[a[1, 2]][a[2, 1]]}
```

$\{0, -a[2,3]\,b_1 + a[1,3]\,b_2, a[2,1]\,b_1 - a[1,2]\,b_2\}$

The Burau representation

```
Burau_n_[x_] := MatrixExp[burau_n[x]]
```

```
Burau_3[a[1, 2]] // MatrixForm
```

$$\begin{pmatrix} 1 & 1-e^{b_1} & 0 \\ 0 & e^{b_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm /@ Simplify@{
   Burau_3[a[1, 2]].Burau_3[a[1, 3]].Burau_3[a[2, 3]],
   Burau_3[a[2, 3]].Burau_3[a[1, 3]].Burau_3[a[1, 2]]
  }
```

$$\left\{\begin{pmatrix} 1 & 1-e^{b_1} & 1-e^{b_1} \\ 0 & e^{b_1} & -e^{b_1}\left(-1+e^{b_2}\right) \\ 0 & 0 & e^{b_1+b_2} \end{pmatrix}, \begin{pmatrix} 1 & 1-e^{b_1} & 1-e^{b_1} \\ 0 & e^{b_1} & -e^{b_1}\left(-1+e^{b_2}\right) \\ 0 & 0 & e^{b_1+b_2} \end{pmatrix}\right\}$$

The Adjoint representation

```
Ad_n_[x_][y_] := deburau[Burau_n[x].burau_n[y].Inverse[Burau_n[x]]];
```

```
(# → Simplify@Ad_4[a[1, 2]][#]) & /@
  {a[3, 4], a[1, 3], a[3, 2], a[3, 1], a[2, 3], a[2, 1]} // Column
```

$a[3,4] \to a[3,4]$

$a[1,3] \to a[1,3]$

$a[3,2] \to \frac{e^{-b_1}\left(a[3,2]\,b_1 + \left(-1+e^{b_1}\right) a[1,2]\,b_3\right)}{b_1}$

$a[3,1] \to a[3,1] + \left(1-e^{-b_1}\right) a[3,2] - \frac{e^{-b_1}\left(-1+e^{b_1}\right) a[1,2]\,b_3}{b_1}$

$a[2,3] \to e^{b_1} a[2,3] - \frac{\left(-1+e^{b_1}\right) a[1,3]\,b_2}{b_1}$

$a[2,1] \to \frac{e^{b_1} a[2,1]\,b_1 - \left(-1+e^{b_1}\right) a[1,2]\,b_2}{b_1}$