Pensieve header: Scatter and Glow in OneCo. Continues pensieve://2016-02/, continued pensieve://Projects/OneCo-1604/.

In the U(T)U(H) conventions.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2016-03"]
```

```
C:\drorbn\AcademicPensieve\2016-03
```

I ought to be able to replace c[i] with a[c,i], and ca with aa, everywhere!

Signs

```
SetSigns[0] := Do[Unset[ϵi], {i, 0, 48}];
```

$$\text{SetSigns}[4] := \Big(\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9, \epsilon_{10}, \epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{15}, \epsilon_{16}, \epsilon_{17}, \epsilon_{18}, \epsilon_{19}, \epsilon_{20}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24}, \epsilon_{25}, \epsilon_{26}, \epsilon_{27}, \epsilon_{28}, \epsilon_{29}, \epsilon_{30}, \epsilon_{31}, \epsilon_{32}, \epsilon_{33}, \epsilon_{34}, \epsilon_{35}, \epsilon_{36}, \epsilon_{37}, \epsilon_{38}, \epsilon_{39}, \epsilon_{40}, \epsilon_{41}, \epsilon_{42}, \epsilon_{43}, \epsilon_{44}, \epsilon_{45}, \epsilon_{46}, \epsilon_{47}, \epsilon_{48}\} = \{\epsilon_1, \epsilon_1, \epsilon_1\epsilon_{10}, \epsilon_5, \epsilon_{10}, \epsilon_5, \epsilon_6, -\epsilon_1\epsilon_5, \epsilon_5\epsilon_9, \epsilon_9, \epsilon_{10}, \epsilon_1\epsilon_{10}, \epsilon_1\epsilon_{10}, -\epsilon_1\epsilon_{10}, \epsilon_1\epsilon_{10}, \epsilon_1\epsilon_{10}, -\epsilon_1\epsilon_9\epsilon_{10}, \epsilon_1\epsilon_9\epsilon_{10}, \frac{\epsilon_1^2\epsilon_5\epsilon_{10}}{\epsilon_6}, \epsilon_1\epsilon_5, \epsilon_1\epsilon_5, \epsilon_1^2\epsilon_5, \epsilon_1\epsilon_5, \epsilon_1^2\epsilon_{10}, \epsilon_1\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \frac{\epsilon_1^2\epsilon_5\epsilon_{10}}{\epsilon_6}, \epsilon_1\epsilon_5, \epsilon_1^2\epsilon_5\epsilon_{10}, \epsilon_1^2\epsilon_5\epsilon_{10}, \epsilon_1^2\epsilon_5\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_5, \epsilon_1\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}^2, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^3\epsilon_{10}^2, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}, \epsilon_1^2\epsilon_{10}\} \Big);$$

$$\text{SetSigns}[5] := \Big(\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9, \epsilon_{10}, \epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{15}, \epsilon_{16}, \epsilon_{17}, \epsilon_{18}, \epsilon_{19}, \epsilon_{20}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24}, \epsilon_{25}, \epsilon_{26}, \epsilon_{27}, \epsilon_{28}, \epsilon_{29}, \epsilon_{30}, \epsilon_{31}, \epsilon_{32}, \epsilon_{33}, \epsilon_{34}, \epsilon_{35}, \epsilon_{36}, \epsilon_{37}, \epsilon_{38}, \epsilon_{39}, \epsilon_{40}, \epsilon_{41}, \epsilon_{42}, \epsilon_{43}, \epsilon_{44}, \epsilon_{45}, \epsilon_{46}, \epsilon_{47}, \epsilon_{48}\} = \{1, 1, \epsilon_{10}, \epsilon_5, \epsilon_{10}, \epsilon_5, \epsilon_6, -\epsilon_5, \epsilon_5\epsilon_9, \epsilon_9, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, -\epsilon_{10}, \epsilon_{10}, \epsilon_{10}, -\epsilon_9\epsilon_{10}, \epsilon_9\epsilon_{10}, \frac{\epsilon_5\epsilon_{10}}{\epsilon_6}, \epsilon_5, \epsilon_5, \epsilon_5, \epsilon_5, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \frac{\epsilon_5\epsilon_{10}}{\epsilon_6}, \epsilon_5, \epsilon_5\epsilon_{10}, \epsilon_5\epsilon_{10}, \epsilon_5\epsilon_{10}, \epsilon_{10}, \epsilon_5, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}^2, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}^2, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}, \epsilon_{10}\} \Big);$$

```
Cases[SetSigns[5], ϵ_, ∞] // Union
```

$\{\epsilon_5, \epsilon_6, \epsilon_9, \epsilon_{10}\}$

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];
(* tests for non-strict ordering. Also true when {is} is {i,i}. *)
Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];

Simp[expr_] := Expand[expr];
S[expr_] :=
  expr /. (λ_β | λ_a | λ_δβ | λ_δa | λ_c | λ_ca | λ_δaa) ⧴ MapAt[Simp, λ, 1];

AutoCollecting[λ_] := (
   λ /: λ[0, ___] = 0;
   λ /: λ[f_, r___] + λ[g_, r___] := λ[Simp[f + g], r];
   λ /: g_ * λ[f_, r___] := λ[Simp[g f], r];
  );
AutoCollecting /@ {β, a, δβ, c, δa, ca, δaa};
UU /: UU[x_] + UU[y_] := UU[x + y];
UU /: a_ * UU[x_] := UU[Expand[a x]];

γ[f_, j_, k_] := δa[f, j, k] - c[ϵ0 bj f, k];
γa[f_, j_, k_, l_, m_] := δaa[f, j, k, l, m] - ca[ϵ0 bj f, k, l, m];
```

Bases

```
UUBasis[T_List, H_List, f_] := Module[
   {ff, n = 0, h, t, h1, h2},
   ff := f++n @@ Table[bt, {t, T}];
   UU /@ Flatten@{
       β[ff],
       Table[{a[ff, t, h], δa[ff, t, h]}, {t, T}, {h, H}],
       δβ[ff],
       Table[c[ff, h], {h, H}],
       Table[ca[ff, h1, t, h2], {h1, H}, {t, T}, {h2, H}],
       Table[δaa[ff, T⟦i⟧, H⟦j⟧, T⟦k⟧, H⟦l⟧],
        {k, Length@T}, {i, k}, {l, Length@H}, {j, l}]
      } /. 1_[___] → 1
  ];
UUBasis[S_List, f_] := UUBasis[S, S, f];
UUBasis[n_Integer, m_Integer, f_] := UUBasis[Range@n, Range@m, f];
UUBasis[n_Integer, f_] := UUBasis[Range@n, f];
```

δaa relations

Switch from thth to tthh indexing? (not for the moment)

```
UU[expr_] // S := UU[S[expr]]; (* Temporary fixture! *)
```

```
UU[expr_] // S := UU[S[expr //. {
    δaa[f_, i_, j_, k_, l_] /; !OQ[j, l] :→ δaa[f, k, l, i, j],
    δaa[f_, i_, j_, k_, l_] /; !OQ[i, k] ⋀ DQ[j, l] ⋀ OQ[j, l] :→
     δaa[f, i, l, k, j] + ca[ϵ1 bk f, l, i, j] + ca[-ϵ1 bi f, l, k, j] +
      ca[-ϵ1 bk f, j, i, l] + ca[ϵ1 bi f, j, k, l],
    δaa[f_, i_, k_, j_, k_] /; !OQ[i, j] :→ δaa[f, j, k, i, k] +
      δa[-ϵ2 bi f, j, k] + δa[ϵ2 bj f, i, k]
   }]];
```

tm, hm, hts, dm

```
UU[expr_] // tm[x_, y_, z_] := S[UU[Expand[expr /. {
     a[f_, x, j_] :→ a[f, z, j] + ϵ3 γ[∂by f, z, j],
     a[f_, y, j_] :→ a[f, z, j],
     δa[f_, x | y, j_] :→ δa[f, z, j],
     ca[f_, i_, x | y, j_] :→ ca[f, i, z, j],
     δaa[f_, i_, j_, k_, l_] :→
      δaa[f, i // Replace[x | y → z], j, k // Replace[x | y → z], l]
    } /. bx|y → bz]]];
```

```
UU[expr_] // hm[x_, y_, z_] := S[UU[Expand[expr /. {
    a[f_, i_, x | y] :→ a[f, i, z],
    c[f_, x | y] :→ c[f, z],
    δa[f_, i_, x | y] :→ δa[f, i, z],
    ca[f_, y, j_, x] :→ ca[f, z, j, z] + ϵ4 γ[f, j, z],
    ca[f_, i_, j_, k_] :→
     ca[f, i // Replace[x | y → z], j, k // Replace[x | y → z]],
    δaa[f_, i_, y, k_, x] :→ δaa[f, k, z, i, z],
    δaa[f_, i_, j_, k_, l_] :→
     δaa[f, i, j // Replace[x | y → z], k, l // Replace[x | y → z]]
   }]]];
```

```
UU[expr_] // hts[y_, x_] := S[UU[Expand[expr /. {
      a[f_, i_, j_] :→ a[f, i, j] - ϵ5 Kδj,y γ[∂bx f, i, y] -
        Kδi,x Kδj,y (ϵ6 β[f bx] + ϵ7 c[f, y] - ϵ8 δβ[bx ∂bx f]),
      δa[f_, x, y] :→ δa[f, x, y] - ϵ9 δβ[f bx],
      ca[f_, i_, j_, k_] :→
       ca[f, i, j, k] + ϵ10 Kδi,y Kδj,x γ[f, x, k] + Kδj,x Kδk,y c[-ϵ11 f bx, i],
      δaa[f_, i_, j_, k_, l_] :→ δaa[f, i, j, k, l] + ϵ12 Kδi,x Kδj,y δa[-bx f, k, l] +
        ϵ13 Kδi,x Kδl,y (-δa[bk f, x, j] + δa[bx f, k, j]) +
        ϵ14 Kδk,x Kδj,y (δa[bi f, x, l] - δa[bx f, i, l]) + ϵ15 Kδk,x Kδl,y δa[-bx f, i, j] +
        ϵ16 Kδi,x Kδj,l,y δβ[bx bk f] + 2 ϵ17 Kδx,i,k Kδy,j,l δβ[bx bx f]
     }]]];

Table[i -> ϵi, {i, 5, 17}]

{5 → ϵ5, 6 → ϵ6, 7 → -ϵ5, 8 → ϵ5 ϵ9, 9 → ϵ9, 10 → ϵ10, 11 → ϵ10,
 12 → ϵ10, 13 → -ϵ10, 14 → ϵ10, 15 → ϵ10, 16 → -ϵ9 ϵ10, 17 → ϵ9 ϵ10}

dm[x_, y_, z_][expr_] := expr // hts[x, y] // tm[x, y, z] // hm[x, y, z]
```

tσ, hσ, dσ on {β, a, δβ, c, δa, ca, δaa}

```
tσ[x_List, y_List][expr_] := Module[{rule = Thread[x → y]},
   S[expr /. bi_ :→ bi /. rule /. {
      a[f_, i_, j_] :→ a[f, i /. rule, j],
      δa[f_, i_, j_] :→ δa[f, i /. rule, j],
      ca[f_, i_, j_, k_] :→ ca[f, i, j /. rule, k],
      δaa[f_, i_, j_, k_, l_] :→ δaa[f, i /. rule, j, k /. rule, l]
     }]
   ];
tσ[x_, y_][expr_] := tσ[{x}, {y}][expr];
hσ[x_List, y_List][expr_] := Module[{rule = Thread[x → y]},
   S[expr /. {
      a[f_, i_, j_] :→ a[f, i, j /. rule],
      c[f_, i_] :→ c[f, i /. rule],
      δa[f_, i_, j_] :→ δa[f, i, j /. rule],
      ca[f_, i_, j_, k_] :→ ca[f, i /. rule, j, k /. rule],
      δaa[f_, i_, j_, k_, l_] :→ δaa[f, i, j /. rule, k, l /. rule]
     }]
   ];
hσ[x_, y_][expr_] := hσ[{x}, {y}][expr];
dσ[x_, y_][expr_] := expr // tσ[x, y] // hσ[x, y];
```

tb, hb, thb, htb, db, bb on {β, a, δβ, c, δa, ca, δaa}

```
tb[x_][UU[L_], UU[R_]] := Module[{p}, S[UU[Expand[Distribute[p[L, R]] /. {
      p[0, _] → 0, p[_, 0] → 0,
      p[_β | _δβ | _c | _δa | _ca | _δaa, _β | _δβ | _c | _δa | _ca | _δaa] → 0,
      p[u_β | u_δβ | u_c | u_δa | u_ca | u_δaa, v_a] :→ -p[v, u]
     } /. {
     p[a[f_, x, j_], u_] :→ (u /. {
         β[g_] :→ ϵ18 γ[f ∂bx g, x, j],
         a[g_, k_, l_] :→ ϵ19 γa[f ∂bx g, x, j, k, l] + Kδx,k (-γa[ϵ20 g ∂bk f, k,
               l, x, j] + ca[ϵ21 f g, l, x, j] - ca[ϵ21 f g, j, k, l]),
         _ → 0
        }),
     p[a[f_, j_, k_], a[g_, x, l_]] /; DQ[j, x] :→ -γa[ϵ22 g ∂bx f, x, l, j, k],
     p[_, _] → 0
    }]]]];

hb[y_][UU[L_], UU[R_]] := Module[{p}, S[UU[Expand[Distribute[p[L, R]] /. {
      p[0, _] → 0, p[_, 0] → 0,
      p[_β | _δβ, _] → 0,
      p[_, _β | _δβ] → 0,
      p[_c | _δa | _ca | _δaa, _c | _δa | _ca | _δaa] → 0,
      p[u_c | u_δa | u_ca | u_δaa, v_a] :→ -p[v, u]
     } /. {
     p[a[f_, i_, y], u_] :→ (u /. {
         a[g_, j_, k_] :→ ϵ23 Kδy,k (a[bj f g, i, y] - a[bi f g, j, k]),
         c[g_, j_] :→ ϵ24 Kδy,j γ[f g, i, j],
         δa[g_, j_, k_] :→ ϵ25 Kδy,k (δa[bj f g, i, y] - δa[bi f g, j, k]),
         ca[g_, j_, k_, l_] :→ Kδy,j γa[ϵ26 f g, i, j, k, l] +
           Kδy,l (ca[ϵ27 bk f g, j, i, y] - ca[ϵ27 bi f g, j, k, l]),
         δaa[g_, j_, k_, l_, m_] :→ ϵ28 Kδy,k (δaa[bj f g, i, y, l, m] - δaa[bi f g, j,
              k, l, m]) + ϵ29 Kδy,m (δaa[bl f g, j, k, i, y] - δaa[bi f g, j, k, l, m])
        }),
     _p → 0
    }]]]];
```

```
thb[x_, y_][UU[L_], UU[R_]] := Module[{p}, S[UU[Expand[Distribute[p[L, R]] /. {
      p[0, _] → 0, p[_, 0] → 0,
      p[_β | _δβ | _c | _δa | _ca | _δaa, _β | _δβ | _c | _δa | _ca | _δaa] → 0,
      p[_a, _β | _δβ] → 0,
      p[β[f_], a[g_, i_, j_]] :→ Kδ_{y,j} γ[ϵ30 g ∂_{b_x} f, i, y],
      p[a[f_, i_, j_], a[g_, k_, l_]] :→ Kδ_{y,l} (
         γa[ϵ31 g ∂_{b_x} f, k, l, i, j] + Kδ_{x,i} (
            γ[-ϵ32 b_k g ∂_{b_x} f, i, j] + δa[ϵ33 b_k g ∂_{b_x} f,
              i, j] - δa[ϵ34 b_i g ∂_{b_x} f, k, j] - a[ϵ35 b_k f g, i, j] + a[
              ϵ35 b_i f g, k, j] + ca[ϵ36 f g, j, k, l] - ca[ϵ36 f g, l, k, j]
           )
        ),
      p[a[f_, i_, j_], c[g_, k_]] :→ -ϵ37 Kδ_{i,x} Kδ_{k,y} γ[f g, i, j],
      p[a[f_, i_, j_], δa[g_, k_, l_]] :→
       ϵ38 Kδ_{x,i} Kδ_{y,l} (-δa[b_k f g, i, j] + δa[b_i f g, k, j]),
      p[a[f_, i_, j_], ca[g_, k_, l_, m_]] :→ Kδ_{x,i} (
         -ϵ39 Kδ_{y,k} γa[f g, i, j, l, m] + ϵ40 Kδ_{y,m}
           (-ca[b_l f g, k, i, j] + ca[b_i f g, k, l, j]) - ϵ41 Kδ_{y,k,m} γ[b_l f g, x, j]
        ),
      p[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :→ Kδ_{x,i} (
         ϵ42 Kδ_{y,l} (-δaa[b_k f g, i, j, m, n] + δaa[b_i f g, k, j, m, n]) +
          ϵ43 Kδ_{y,n} (-δaa[b_m f g, k, l, i, j] + δaa[b_i f g, k, l, m, j]) +
          ϵ44 Kδ_{y,l,n} (δa[b_x b_m f g, k, j] - δa[b_k b_m f g, x, j])
        ),
      p[_δβ | _c, _a] → 0,
      p[δa[f_, i_, j_], a[g_, k_, l_]] :→
       ϵ45 Kδ_{x,i} Kδ_{y,l} (-δa[b_k f g, i, j] + δa[b_i f g, k, j]),
      p[ca[f_, m_, i_, j_], a[g_, k_, l_]] :→ ϵ46 Kδ_{x,i} Kδ_{y,l}
        (-ca[b_k f g, m, i, j] + ca[b_i f g, m, k, j]),
      p[δaa[f_, i_, j_, m_, n_], a[g_, k_, l_]] :→
       ϵ47 Kδ_{x,i} Kδ_{y,l} (-δaa[b_k f g, i, j, m, n] + δaa[b_i f g, k, j, m, n]) +
        ϵ48 Kδ_{x,m} Kδ_{y,l} (-δaa[b_k f g, i, j, m, n] + δaa[b_m f g, i, j, k, n])
     }]]]];
htb[x_, y_][L_UU, R_UU] := -thb[y, x][R, L];
```

$t_1 h_1 t_2 h_2 \to t_1 t_2 h_1 h_2 \to t_2 t_1 h_1 h_2 \to t_2 t_1 h_2 h_1 \to t_2 h_2 t_1 h_1$:

```
db[x_][u_UU, v_UU] := Module[{t, h}, Plus[
   htb[x, x][u // tσ[x, t], v // hσ[x, h]] // tm[t, x, x] // hm[x, h, x],
   tb[x][u, v // hσ[x, h]] // hm[x, h, x],
   hb[x][u, v // tσ[x, t]] // tm[t, x, x],
   thb[x, x][u // hσ[x, h], v // tσ[x, t]] // tm[t, x, x] // hm[x, h, x]
  ]];
```

```
bb[S_List] := Module[{w, bar, t, n = 0},
   bar[x_] := -x;
   w = #2 // dσ[S, bar /@ S];
   Sum[
    t = db[S⟦k⟧][#1, w // dσ[bar[S⟦k⟧], S⟦k⟧]];
    Do[t = t // dm[bar[S⟦i⟧], S⟦i⟧, S⟦i⟧], {i, 1, k - 1}];
    Do[t = t // dm[S⟦i⟧, bar[S⟦i⟧], S⟦i⟧], {i, k + 1, Length@S}];
    t,
    {k, Length@S}
   ]
  ] &
bb[S___] := bb[{S}]
```

ct (contract)

```
ct::usage =
  "ct[h,t][L,R] contracts the head h in L with the tail t in R. ct[s][L,R] takes
    h=t=s, and ct[][L,R] takes s=0. When ambiguous, L is placeed below R.";
ct[s_] := ct[s, s];
ct[] = ct[0, 0];
ct[h_, t_][UU[L_], UU[R_]] := Module[{p}, S[UU[Distribute[p[L, R]] /. {
        p[_β | _δβ, _] → 0,
        p[a[f_, i_, h], β[g_]] :→ β[f b_i ((∂_{b_t} g) /. b_t → 0)],
        p[a[f_, i_, h], a[g_, t, j_]] :→ a[f (g /. b_t → 0), i, j],
        p[a[f_, i_, h], a[g_, j_, k_]] :→ a[f b_i ((∂_{b_t} g) /. b_t → 0), j, k],
        p[a[f_, i_, h], c[g_, j_]] :→ c[f b_i ((∂_{b_t} g) /. b_t → 0), j],
        p[a[f_, i_, h], δa[g_, t, j_]] :→ δa[f (g /. b_t → 0), i, j],
        p[a[f_, i_, h], δa[g_, j_, k_]] :→ δa[f b_i ((∂_{b_t} g) /. b_t → 0), j, k],
        p[a[f_, i_, h], ca[g_, k_, t, j_]] :→ ca[f (g /. b_t → 0), k, i, j],
        p[a[f_, i_, h], ca[g_, l_, j_, k_]] :→ ca[f b_i ((∂_{b_t} g) /. b_t → 0), l, j, k],
        p[a[f_, i_, h], δaa[g_, t, j_, t, k_]] → 0,
        p[a[f_, i_, h], δaa[g_, t, j_, k_, l_]] :→ δaa[f (g /. b_t → 0), i, j, k, l],
        p[a[f_, i_, h], δaa[g_, j_, k_, t, l_]] :→ δaa[f (g /. b_t → 0), j, k, i, l],
        p[a[f_, i_, h], δaa[g_, j_, k_, l_, m_]] :→
         δaa[f b_i ((∂_{b_t} g) /. b_t → 0), j, k, l, m],
        p[a[__], _] → 0,
        p[c[f_, h], β[g_]] :→ δβ[f ((∂_{b_t} g) /. b_t → 0)],
        p[_c, _β] → 0,
        p[c[f_, h], a[g_, t, j_]] :→ c[f (g /. b_t → 0), j],
        p[c[f_, h], a[g_, j_, k_]] :→ δa[f ((∂_{b_t} g) /. b_t → 0), j, k],
        p[_c, _a] → 0,
        p[_c | _δa | _ca | _δaa, _δβ | _c | _δa | _ca | _δaa] → 0,
```

```
p[δa[f_, i_, h], β[g_]] :→ δβ[f b_i ((∂_{b_t} g) /. b_t → 0)],
p[δa[f_, i_, h], a[g_, t, j_]] :→ δa[f (g /. b_t → 0), i, j],
p[δa[f_, i_, h], a[g_, j_, k_]] :→ δa[f b_i ((∂_{b_t} g) /. b_t → 0), j, k],
p[_δa, _] → 0,
p[ca[_, h, _, h], _] → 0,
p[ca[f_, h, i_, j_], β[g_]] :→ δa[f ((∂_{b_t} g) /. b_t → 0), i, j],
p[ca[f_, i_, j_, h], β[g_]] :→ c[f b_j ((∂_{b_t} g) /. b_t → 0), i],
p[ca[f_, h, i_, j_], a[g_, t, k_]] :→ ca[f (g /. b_t → 0), k, i, j],
p[ca[f_, h, i_, j_], a[g_, k_, l_]] :→ δaa[f ((∂_{b_t} g) /. b_t → 0), i, j, k, l],
p[ca[f_, i_, j_, h], a[g_, t, k_]] :→ ca[f (g /. b_t → 0), i, j, k],
p[ca[f_, i_, j_, h], a[g_, k_, l_]] :→ ca[f b_j ((∂_{b_t} g) /. b_t → 0), i, k, l],
p[_ca, _] → 0,
p[δaa[_, _, h, _, h], _] → 0,
p[δaa[f_, i_, h, j_, k_], β[g_]] :→ δa[f b_i ((∂_{b_t} g) /. b_t → 0), j, k],
p[δaa[f_, i_, h, j_, k_], a[g_, t, l_]] :→ δaa[f (g /. b_t → 0), i, l, j, k],
p[δaa[f_, i_, h, j_, k_], a[g_, l_, m_]] :→
 δaa[f b_i ((∂_{b_t} g) /. b_t → 0), j, k, l, m],
p[δaa[f_, i_, j_, k_, h], β[g_]] :→ δa[f b_k ((∂_{b_t} g) /. b_t → 0), i, j],
p[δaa[f_, i_, j_, k_, h], a[g_, t, l_]] :→ δaa[f (g /. b_t → 0), i, j, k, l],
p[δaa[f_, i_, j_, k_, h], a[g_, l_, m_]] :→
 δaa[f b_k ((∂_{b_t} g) /. b_t → 0), i, j, l, m],
p[_δaa, _] → 0
}]]];
```

dect (de-contract)

```
dect::usage =
  "dect[h,t][uu] returns a pair {L,R} such that ct[h,t][L,R]=uu. Similarly
    for dect[s] and dect[]. uu is assumed to be atomic.";
dect[s_] := dect[s, s];
dect[] = dect[0, 0];
dect[h_, t_][β[f_]] := {};
dect[h_, t_][δβ[f_]] := TBD;
```

Ad

```
AutoAd[B_, x_][y_] :=
  Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
   pows = NestList[B[x, #] &, y, 20];
   Print["pows computed for ", {x, y}, "..."];
   states = Union[Cases[pows,
      s_β | s_δβ | s_a | s_c | s_δa | s_ca | s_δaa :> ReplacePart[s, 1 → _], ∞]];
   UU@Sum[
     seq = Cases[{#}, states[[i]], ∞] & /@ pows;
     seq = Replace[seq, {{_[f_, ___]} :> f, {} → 0}, {1}];
     Print["seq computed... ", states[[i]], " is ", i, "/", Length@states];
     dseq = Drop[seq, sh];
     If[Union[Length[MonomialList[#]] & /@ dseq] === {1} ∧
       Union[Length[FactorTermsList[#]] & /@ dseq] === {2},
      sf1 = FindSequenceFunction[FactorTermsList[#][[1]] & /@ dseq];
      sf2 = FindSequenceFunction[FactorTermsList[#][[2]] & /@ dseq];
      Print["sf1: ", sf1, " sf2: ", sf2];
      sf = (sf1[#] sf2[#] &),
      (*Else*) sf = FindSequenceFunction[dseq,
        FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
          "Polynomial", "RationalFunction", "HypergeometricTerm"}];
      Print["sf: ", sf];
     ];
     ReplacePart[states[[i]], 1 → Simplify[Sum[seq[[n + 1]]/n!, {n, 0, sh-1}] + Sum[sf[n + 1 - sh]/n!, {n, sh, ∞}]]],
     {i, Length@states}]
  ];
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)

AutoAd[bb[1, 2], UU@a[1, 1, 2]][UU@a[1, 0, 1]]
```

pows computed for {UU[a[1, 1, 2]], UU[a[1, 0, 1]]}...

seq computed... a[_, 0, 1] is 1/14

sf1: 1 & sf2: 0 &

seq computed... a[_, 0, 2] is 2/14

sf1: $(-1)^{1+\#1}$ & sf2: $b_1^4 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... a[_, 1, 2] is 3/14

sf1: $(-1)^{\#1}$ & sf2: $b_0 b_1^3 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... c[_, 2] is 4/14

sf1: $(-1)^{\#1}$ & sf2: $b_0 b_1^3 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 1, 0, 2] is 5/14

sf1: $(-1)^{\#1}$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 1, 1, 2] is 6/14

sf1: $(-1)^{1+\#1}$ & sf2: $b_0 b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 2, 0, 1] is 7/14

sf1: $(-1)^{1+\#1}$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 2, 0, 2] is 8/14

sf1: $(-1)^{1+\#1} (-5 + 2^{4+\#1} - \#1)$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 2, 1, 2] is 9/14

sf1: $(-1)^{\#1} (-9 + 2^{4+\#1} - 2\,\#1)$ & sf2: $b_0 b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δa[_, 0, 2] is 10/14

sf1: $(-1)^{1+\#1} (-4 + 2^{4+\#1} - \#1)$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δa[_, 1, 2] is 11/14

sf1: $(-1)^{\#1} (-5 + 2^{4+\#1} - \#1)$ & sf2: $b_0 b_1^2 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δaa[_, 0, 1, 1, 2] is 12/14

sf1: $(-1)^{\#1}$ & sf2: $b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δaa[_, 0, 2, 1, 2] is 13/14

sf1: $(-1)^{\#1} (-4 + 2^{4+\#1} - \#1)$ & sf2: $b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δaa[_, 1, 2, 1, 2] is 14/14

sf1: $-2 (-1)^{\#1} (-4 + 2^{3+\#1} - \#1)$ & sf2: $b_0 b_1 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

$$\text{UU}\Big[\text{a}[1,0,1]+\text{a}\big[1-e^{-b_1\epsilon_{10}},0,2\big]+\text{a}\Big[\frac{\left(-1+e^{-b_1\epsilon_{10}}\right)b_0}{b_1},1,2\Big]+\text{c}\Big[b_0\,\epsilon_5\left(\frac{-1+e^{-b_1\epsilon_{10}}}{b_1}+\epsilon_{10}\right),2\Big]+$$
$$\text{ca}\Big[-\frac{\left(1-e^{-b_1\epsilon_{10}}\right)\epsilon_5}{b_1\,\epsilon_{10}},1,0,2\Big]+\text{ca}\Big[\frac{\left(1-e^{-b_1\epsilon_{10}}\right)\epsilon_5}{b_1\,\epsilon_{10}},2,0,1\Big]+$$
$$\text{ca}\Big[\frac{b_0\,\epsilon_5\left(1-e^{-b_1\epsilon_{10}}-b_1\,\epsilon_{10}\right)}{b_1^2\,\epsilon_{10}},1,1,2\Big]+\text{ca}\Big[-\frac{e^{-2b_1\epsilon_{10}}\,\epsilon_5\left(1-e^{b_1\epsilon_{10}}+e^{b_1\epsilon_{10}}\,b_1\,\epsilon_{10}\right)}{b_1\,\epsilon_{10}},2,0,2\Big]+$$
$$\text{ca}\Big[-\frac{e^{-2b_1\epsilon_{10}}\,b_0\,\epsilon_5\left(-1+e^{b_1\epsilon_{10}}+e^{b_1\epsilon_{10}}\left(-2+e^{b_1\epsilon_{10}}\right)b_1\,\epsilon_{10}\right)}{b_1^2\,\epsilon_{10}},2,1,2\Big]+$$
$$\delta\text{a}\Big[\epsilon_5\left(\frac{1-e^{-2b_1\epsilon_{10}}}{b_1}+\left(-1-e^{-b_1\epsilon_{10}}\right)\epsilon_{10}\right),0,2\Big]+$$
$$\delta\text{a}\Big[\frac{e^{-2b_1\epsilon_{10}}\,b_0\,\epsilon_5\left(1-e^{b_1\epsilon_{10}}+e^{b_1\epsilon_{10}}\,b_1\,\epsilon_{10}\right)}{b_1^2},1,2\Big]+\delta\text{aa}\Big[\frac{\epsilon_5\left(b_1+\frac{-1+e^{-b_1\epsilon_{10}}}{\epsilon_{10}}\right)}{b_1^2},0,1,1,2\Big]+$$
$$\delta\text{aa}\Big[\frac{e^{-2b_1\epsilon_{10}}\,b_0\,\epsilon_5\left(-1+e^{2b_1\epsilon_{10}}-2\,e^{b_1\epsilon_{10}}\,b_1\,\epsilon_{10}\right)}{b_1^3\,\epsilon_{10}},1,2,1,2\Big]+$$
$$\delta\text{aa}\Big[\frac{e^{-2b_1\epsilon_{10}}\left(1+e^{b_1\epsilon_{10}}\right)\epsilon_5\left(1-e^{b_1\epsilon_{10}}+e^{b_1\epsilon_{10}}\,b_1\,\epsilon_{10}\right)}{b_1^2\,\epsilon_{10}},0,2,1,2\Big]\Big]$$

```
AutoAd[bb[1, 2], UU@a[1, 1, 2]][UU@a[1, 0, 2]]
```

```
pows computed for {UU[a[1, 1, 2]], UU[a[1, 0, 2]]}...
seq computed... a[_, 0, 2] is 1/8
```

sf1: $(-1)^{\sharp 1}$ & sf2: $b_1^4 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... a[_, 1, 2] is 2/8

sf1: $(-1)^{1+\sharp 1}$ & sf2: $b_0 b_1^3 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... ca[_, 2, 0, 2] is 3/8

sf1: $(-1)^{\sharp 1} (-5 + 2^{4+\sharp 1} - \sharp 1)$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... ca[_, 2, 1, 2] is 4/8

sf1: $-2 (-1)^{\sharp 1} (-4 + 2^{3+\sharp 1} - \sharp 1)$ & sf2: $b_0 b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... δa[_, 0, 2] is 5/8

sf1: $(-1)^{\sharp 1} (-5 + 2^{4+\sharp 1} - \sharp 1)$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... δa[_, 1, 2] is 6/8

sf1: $(-1)^{1+\sharp 1} (-5 + 2^{4+\sharp 1} - \sharp 1)$ & sf2: $b_0 b_1^2 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... δaa[_, 0, 2, 1, 2] is 7/8

sf1: $(-1)^{1+\sharp 1} (-5 + 2^{4+\sharp 1} - \sharp 1)$ & sf2: $b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\sharp 1}$ &

seq computed... δaa[_, 1, 2, 1, 2] is 8/8

sf1: $2 (-1)^{\sharp 1} (-4 + 2^{3+\sharp 1} - \sharp 1)$ & sf2: $b_0 b_1 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\sharp 1}$ &

$$\mathrm{UU}\Big[\mathrm{a}\big[e^{-b_1 \epsilon_{10}}, 0, 2\big] + \mathrm{a}\Big[\frac{\left(1 - e^{-b_1 \epsilon_{10}}\right) b_0}{b_1}, 1, 2\Big] +$$
$$\mathrm{ca}\Big[\frac{e^{-2 b_1 \epsilon_{10}} b_0 \epsilon_5 \left(-1 + e^{2 b_1 \epsilon_{10}} - 2 e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^2 \epsilon_{10}}, 2, 1, 2\Big] +$$
$$\mathrm{ca}\Big[\frac{e^{-2 b_1 \epsilon_{10}} \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1 \epsilon_{10}}, 2, 0, 2\Big] +$$
$$\delta\mathrm{a}\Big[-\frac{e^{-2 b_1 \epsilon_{10}} b_0 \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^2}, 1, 2\Big] +$$
$$\delta\mathrm{a}\Big[\frac{e^{-2 b_1 \epsilon_{10}} \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1}, 0, 2\Big] +$$
$$\delta\mathrm{aa}\Big[-\frac{e^{-2 b_1 \epsilon_{10}} b_0 \epsilon_5 \left(-1 + e^{2 b_1 \epsilon_{10}} - 2 e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^3 \epsilon_{10}}, 1, 2, 1, 2\Big] +$$
$$\delta\mathrm{aa}\Big[-\frac{e^{-2 b_1 \epsilon_{10}} \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^2 \epsilon_{10}}, 0, 2, 1, 2\Big]\Big]$$

```
AutoAd[bb[1, 2], UU@a[1, 1, 2]][UU@a[1, 1, 0]]
```

```
pows computed for {UU[a[1, 1, 2]], UU[a[1, 1, 0]]}...
seq computed... a[_, 1, 0] is 1/4
sf1: 1 & sf2: 0 &
seq computed... ca[_, 0, 1, 2] is 2/4
sf1: 1 & sf2: 0 &
seq computed... ca[_, 2, 1, 0] is 3/4
```

sf1: $(-1)^{\#1}$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δaa[_, 1, 0, 1, 2] is 4/4

sf1: $(-1)^{1+\#1}$ & sf2: $b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

$$\text{UU}\Big[\text{a}[1, 1, 0] + \text{ca}[\epsilon_5, 0, 1, 2] + \text{ca}\Big[-\frac{\left(1 - e^{-b_1 \epsilon_{10}}\right) \epsilon_5}{b_1 \epsilon_{10}}, 2, 1, 0\Big] + \delta\text{aa}\Big[-\frac{\epsilon_5 \left(-1 + e^{-b_1 \epsilon_{10}} + b_1 \epsilon_{10}\right)}{b_1^2 \epsilon_{10}}, 1, 0, 1, 2\Big]\Big]$$

AutoAd[bb[1, 2], UU@a[1, 1, 2]][UU@a[1, 2, 0]]

```
pows computed for {UU[a[1, 1, 2]], UU[a[1, 2, 0]]}...
seq computed... a[_, 1, 0] is 1/9
```
sf1: -1 & sf2: $b_1^3 b_2 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... a[_, 2, 0] is 2/9

sf1: 1 & sf2: $b_1^4 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... c[_, 0] is 3/9

sf1: $3+\#1$ & sf2: $b_1^3 b_2 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 0, 1, 2] is 4/9

sf: $-b_1^2 (b_1 - 2 b_2 - \#1 b_2) \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 0, 2, 2] is 5/9

sf1: $-3-\#1$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... ca[_, 2, 1, 0] is 6/9

sf1: $(-1)^{1+\#1}$ & sf2: $b_1^3 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δa[_, 1, 0] is 7/9

sf1: $-3-\#1$ & sf2: $b_1^2 b_2 \epsilon_5 \epsilon_{10}^4 (b_1 \epsilon_{10})^{\#1}$ &

seq computed... δaa[_, 1, 0, 1, 2] is 8/9

sf: $b_1 \epsilon_5 \epsilon_{10}^3 \left(b_1 (-b_1 \epsilon_{10})^{\#1} + b_1 (b_1 \epsilon_{10})^{\#1} - 2 b_2 (b_1 \epsilon_{10})^{\#1} - \#1\, b_2 (b_1 \epsilon_{10})^{\#1}\right)$ &

seq computed... δaa[_, 1, 0, 2, 2] is 9/9

sf1: $3+\#1$ & sf2: $b_1^2 \epsilon_5 \epsilon_{10}^3 (b_1 \epsilon_{10})^{\#1}$ &

$$\begin{aligned}&\text{UU}\Big[a\big[e^{b_1 \epsilon_{10}}, 2, 0\big] + a\Big[-\frac{\left(-1+e^{b_1 \epsilon_{10}}\right) b_2}{b_1}, 1, 0\Big] + c\Big[\frac{b_2 \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1}, 0\Big] +\\
&\quad ca\Big[\frac{\left(1-e^{-b_1 \epsilon_{10}}\right) \epsilon_5}{b_1 \epsilon_{10}}, 2, 1, 0\Big] + ca\Big[-\frac{\epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1 \epsilon_{10}}, 0, 2, 2\Big] +\\
&\quad ca\Big[\frac{\epsilon_5 \left(-2\left(-1+e^{b_1 \epsilon_{10}}\right) b_2 + b_1 \left(1 - e^{b_1 \epsilon_{10}} + \left(1+e^{b_1 \epsilon_{10}}\right) b_2 \epsilon_{10}\right)\right)}{b_1^2 \epsilon_{10}}, 0, 1, 2\Big] +\\
&\quad \delta a\Big[-\frac{b_2 \epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^2}, 1, 0\Big] +\\
&\quad \delta aa\Big[\frac{\epsilon_5 \left(1 - e^{b_1 \epsilon_{10}} + e^{b_1 \epsilon_{10}} b_1 \epsilon_{10}\right)}{b_1^2 \epsilon_{10}}, 1, 0, 2, 2\Big] + \delta aa\Big[-\frac{1}{b_1^3 \epsilon_{10}} e^{-b_1 \epsilon_{10}} \epsilon_5\\
&\qquad \left(-2 e^{b_1 \epsilon_{10}} \left(-1+e^{b_1 \epsilon_{10}}\right) b_2 + b_1 \left(-\left(-1+e^{b_1 \epsilon_{10}}\right)^2 + e^{b_1 \epsilon_{10}} \left(1+e^{b_1 \epsilon_{10}}\right) b_2 \epsilon_{10}\right)\right), 1, 0, 1, 2\Big]\Big]\end{aligned}$$