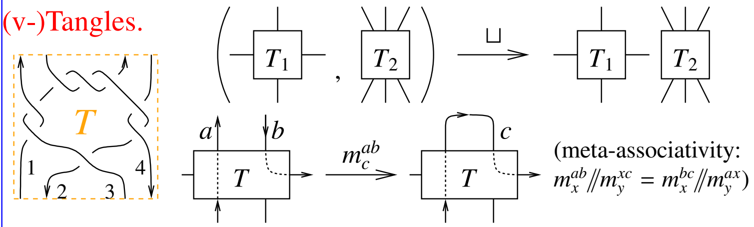
Work in Progress on **Polynomial Time Knot Polynomials, A**

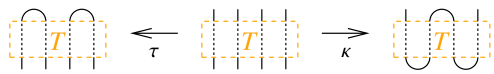
Abstrant. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, mostly run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



(meta-associativity: $m^{ab}_x /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$)

Why Tangles?



- Finitely presented.
- Divide and conquer computations.
- "Alg. Knot Theory": If K is ribbon, $z(K) \in \{\kappa(\zeta)\colon \tau(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).

$U \in \mathcal{T}_n \xleftarrow{\tau} \mathcal{T}_{2n} \xrightarrow{\kappa}$ ribbon $K \in \mathcal{T}_1$; $\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$; $1 \in \mathcal{A}_n \xleftarrow{\tau} \mathcal{A}_{2n} \xrightarrow{\kappa} z(K) \in \mathcal{A}_1$

Faster is better, leaner is meaner!

Theorem 1.

∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\\hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\\hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\\hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\\hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\substack{T_a, T_b \to T_c\\ \mu := 1-\beta}]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\\hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\!\nearrow_b,\ {}_b\!\nwarrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\\hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\\hline a & 1 & 1-T_a^{\pm 1}\\ b & 0 & T_a^{\pm 1}\end{array}\right)$.

In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ ⧴ a, ∞];
   M = Outer[Factor[∂_{h#1 t#2} λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];

Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{ta,ha}λ ∂_{ta,hb}λ ∂_{ta}λ; ∂_{tb,ha}λ ∂_{tb,hb}λ ∂_{tb}λ; ∂_{ha}λ ∂_{hb}λ λ) /. (t|h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ+αδ/μ ϵ+δθ/μ; ϕ+αψ/μ Ξ+ψθ/μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1-T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1/T_a;
```

Meta-Associativity

Runs.

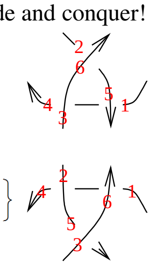
```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```

```
True
```

R3 … divide and conquer!

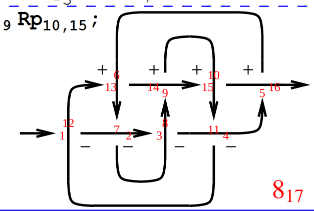
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$

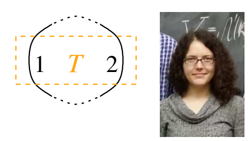


8_{17}

Closed Components.

The Halacheva meta-trace tr_c satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

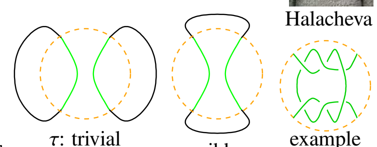
$$\begin{array}{c|cc}\omega & c & S\\\hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\\hline S & \Xi + \psi\theta/\mu\end{array}$$



Halacheva

```
tr_c_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{tc,hc}λ ∂_{tc}λ; ∂_{hc}λ λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

```
True
```



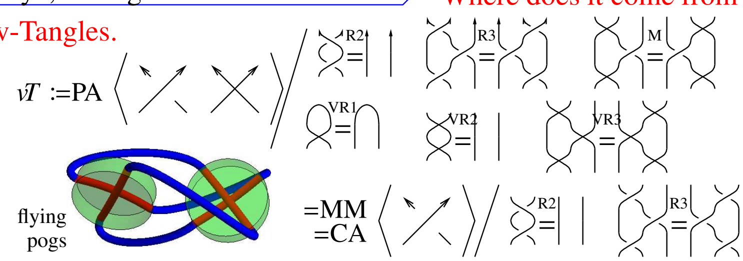
τ: trivial κ: ribbon example

Weaknesses.

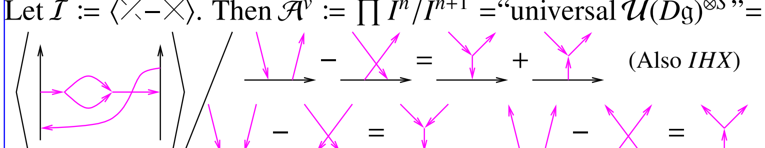
- m^{ab}_c and tr_c are non-linear.
- The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions.
- I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



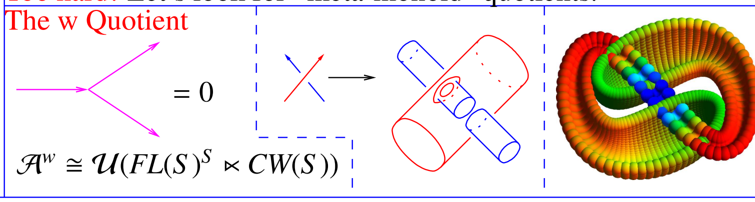
(Also IHX)

Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

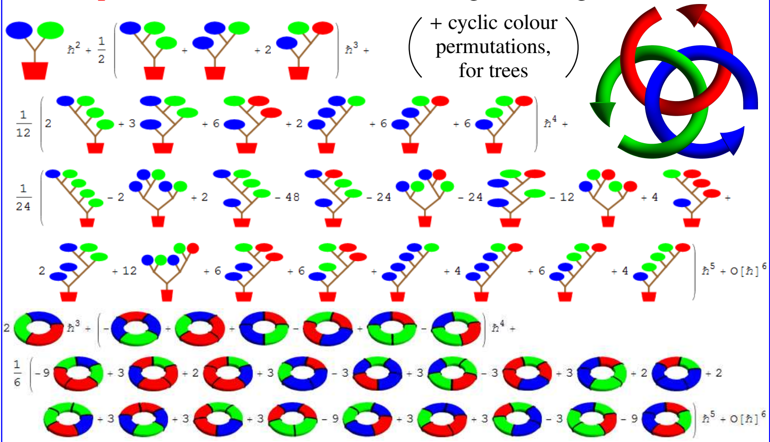
The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

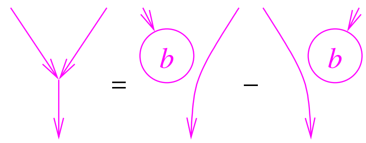
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:



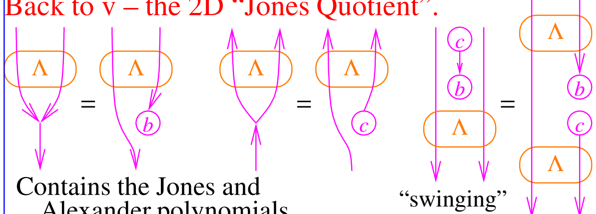
(I have a fancy free-Lie calculator!) (ωεβ/FLD) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.

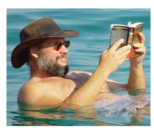


$[u, v] = b_u v - b_v u$

Back to v – the 2D "Jones Quotient".

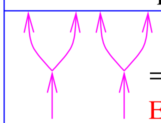


Contains the Jones and Alexander polynomials, ... yet still too hard!



V. Jones

The OneCo Quotient. Likely related to [ADO]



= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

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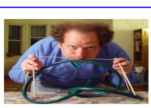


Help Needed

I'm slow and feeble-minded.

The Abstract Context

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

$$\text{meta-associativity:} \quad m^{ab}_x /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$$

$$\text{meta-locality:} \quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:} \quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Theorem. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with

$$\Gamma_1(S) = R_S \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := R_S\langle S\rangle).$$

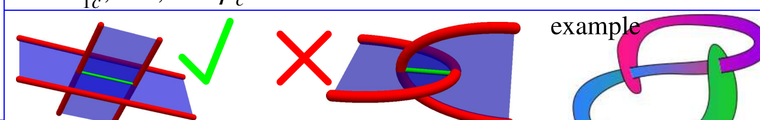
Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c\right)$$

What's done? The braid part, with still-ugly formulas.

What's missing? A lot of concept- and detail-sensitive work towards m^{ab}_{1c}, α^{ab}, and ρ^{ab}_c. The "ribbon element".



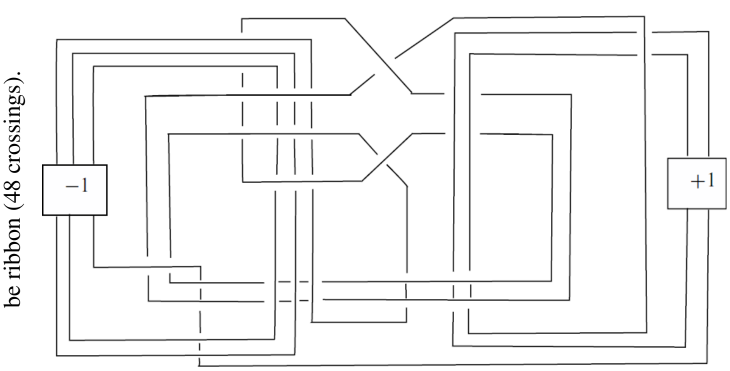
a ribbon singularity a clasp singularity (Eppstein)

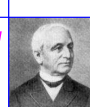
A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

[GST]: a slice knot that might not be ribbon (48 crossings).





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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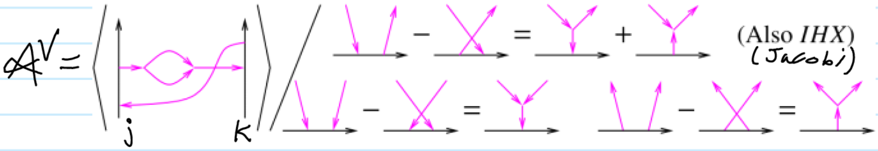
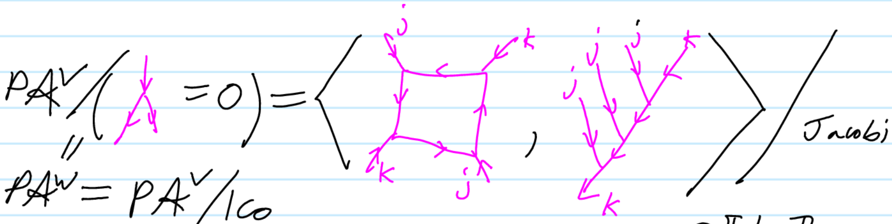
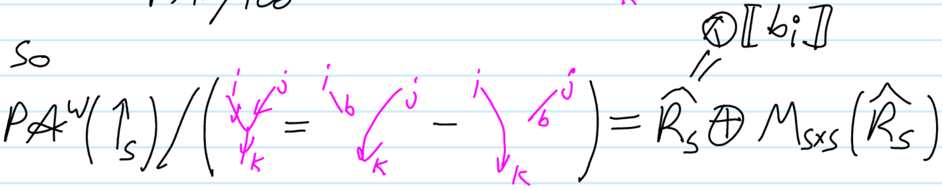
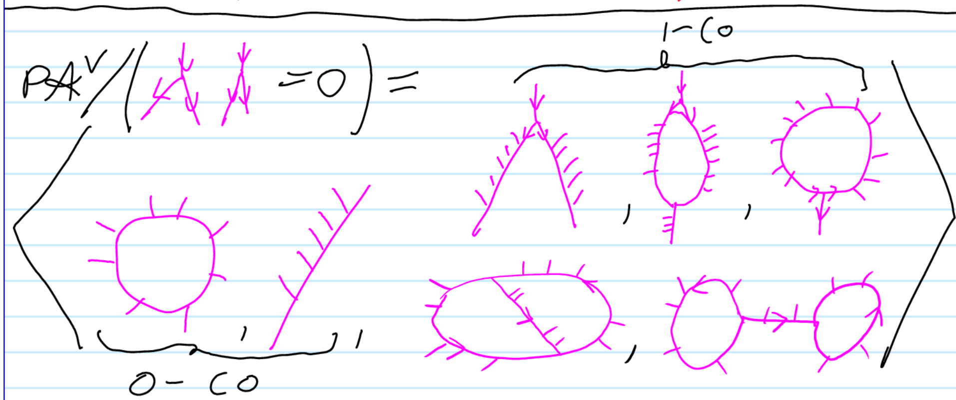


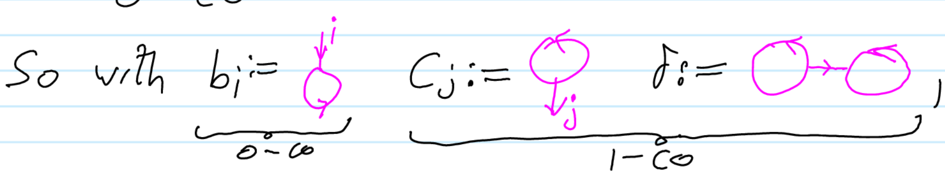


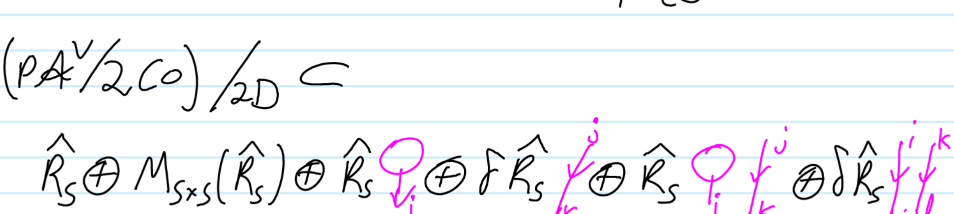
So



and the rest is (hard!) calculations, which lead to a simple rational-function result.







$$= V_S + V_S^{\otimes 2} + V_S + V_S^{\otimes 2} + V_S^{\otimes 3} + (S^2(V_S))^{\otimes 2}$$

[The product law is awful, but experience shows that things simplify....]

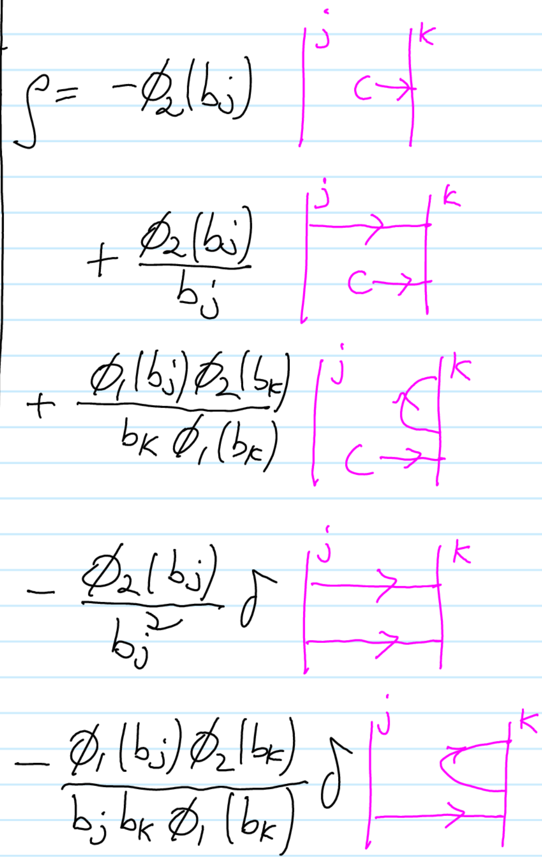
Stitching is clearly possible, but I still don't have explicit formulas.

Proposition The element R_{ij} given below solves the YB equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathcal{A}^v/2co/2D$!

$R_{jk} = e^{j \text{---} k} e^{\rho}$, with



where $\phi_1(x) = e^{-x} - 1$

and $\phi_2(x) = \frac{(x+2)e^{-x} - 2 + x}{2x}$

The Most Important Missing Infrastructure Project in Knot Theory

January-23-12 10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical. This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

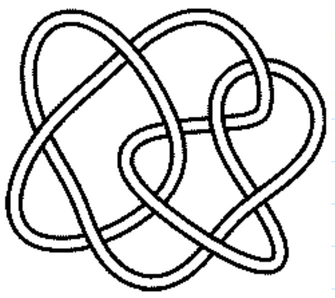
Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

An even better tabulation should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

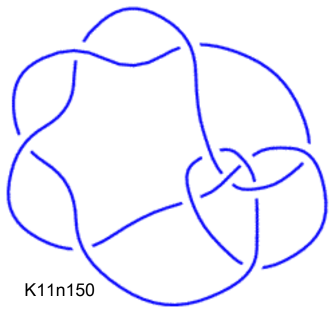
Overall this would be a major project, well worthy of your time.

(Source: http://drorbn.net/AcademicPensieve/2012-01/)



(KnotPlot image)

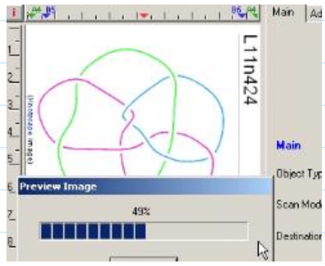
9_42 is Alexander Stoimenow's favourite



(Knotscape image)

The interchange of I-95 and I-695, northeast of Baltimore. (more)

From [AKT-CFA]



From [FastKh]

http://katlas.org/