

*Help Needed!* I wish I could read, or maybe I wish I could write, a survey paper on Taylor expansions for groups with an abstract and a “master table” as below. Oh wait, I’m actually in the process of writing that paper\*! But it’s going hard because I’m underqualified — I don’t know the existing general material well enough, and many of the entries of the “master table” require specialized knowledge that I don’t have.

**Abstract.** First year students learn that the Taylor expansion  $Z_T$  carries functions into power series, and that it has some nice algebraic properties (e.g. multiplicativity,  $Z_T(fg) = Z_T(f)Z_T(g)$ ). It is less well known that the same game can be played within arbitrary groups: there is a natural way to say “a Taylor expansion  $Z$  for elements of an arbitrary group  $G$ ”, and a natural way to carry the algebraic properties of the Taylor expansion to this more gen-

eral context. In the case of a general  $G$  “Taylor expansions” (expansions with the same good properties as  $Z_T$ ) may or may not exist, may or may not be unique, may or may not separate group elements, and a further good property which is hidden in the case of  $Z_T$ , “quadraticity”, may or may not hold.

The purpose of this expository note is to properly define all the notions in the above paragraph, to enumerate some classes of groups whose theory of expansions we either understand or wish to understand, to indicate the relationship between these notions and the notions of “finite type invariants” and “unipotent” and “Mal’cev” completions, and to point out (with references) that our generalization of “expansions” to arbitrary groups is merely the tip of an iceberg, for almost everything we say can be generalized further to “expansions for arbitrary algebraic structures”.

\*See <http://drorbn.net/ExQu>. Also see Suci-Wang, [arXiv:1504.08294](https://arxiv.org/abs/1504.08294).

✓:=Yes, ✗:=No, ?:=Unknown (to the author).  
Superscripts: see “table footnotes” below.

Group(s) $G$	Faithful $Z$ ?	Taylor $Z$ ?	Quadratic?	See
1. Finite / torsion groups	✗ <sup>1</sup>	✓ <sup>2</sup>	✓ <sup>2</sup>	Sec. 2.1
2. Free Abelian groups $\mathbb{Z}^n$	✓	✓	✓	Sec. 1.4.2
3. Free groups $FG_n$	✓	✓	✓	Sec. 1.4.3
4. LOT and LOF groups	✗ <sup>3</sup>	✓	✓	Sec. 4.1
5. Knot and pure tangle groups	✗ <sup>3</sup>	✓	✓	Sec. 4.2
6. Link groups	✗ <sup>4</sup>	✓	✓	Sec. 4.2
7. Pure braid groups $PB_n$	✓	✓	✓	Sec. 1.5
8. Reduced free groups $RF_n$	✓	✓	✗	Sec. 4.3
9. Reduced (homotopy) pure braid groups $RPB_n$	✓	✓	✗	
10. Pure $v$ -braid groups $PvB_n$	?	✗	✓	
11. Pure $w$ -braid groups $PwB_n$	✓	✓	✓	
12. Pure $f$ -braid groups $PfB_n$				Merkov
13. Annular braids				
14. Elliptic pure braid groups $PB_n^1$ (braids on the torus)	?	✓	✗	
15. Higher genus pure braid groups $PB_n^{>1}$ (braids on high genus surfaces)	?	?	✗	<a href="https://arxiv.org/abs/math/0309245">arXiv:math/0309245?</a>
16. Commutators $[G, G]$ in general				
17. Commutators $[PvB_n, PvB_n]$				
18. Commutators $[PvB_n, PvB_n]$				
19. Commutators $[PwB_n, PwB_n]$				
20. Hilden braids				
21. Mexican plait braids				Kurpita-Murasugi
22. Cactus groups				
23. Fundamental groups of surfaces		✓	✓	
24. Mapping class groups				
25. Torelli groups				Hain
26. Right-angled Artin groups		✓	✓	
27. General Artin groups				
28. Groups from BEER				<a href="https://arxiv.org/abs/math/0509661">arXiv:math/0509661</a>
29. Groups from Brochier				<a href="https://arxiv.org/abs/1209.0417">arXiv:1209.0417</a>
30. Poly-free groups				<a href="https://arxiv.org/abs/math/0603470">arXiv:math/0603470</a>

Table 1: Some groups and their expansion properties.

**Table footnotes.** 1. Except  $G = \{e\}$ . 2. In an empty manner. 3. Except  $G = FG_n$ . 4. Except  $G = \mathbb{Z}^n$ .