Seyfaddini: Continuous symplectic topology and area-preserving homeomorphisms

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1. (M,ω) Manifold w/ closed non-deg 2-form
So dim must be $2n$

$Symp(M,\omega) = \{\phi \in Diff(M) : \phi^*\omega = \omega\}$

(in 2D this is "Area preserving")

Example: Hamiltonian flows.

ω allows to assign areas to loops!

(In $\mathbb{R}^4$ this comes out to be the sum of the areas of the projections to $\mathbb{R}^{\{1,2\}}$ & $\mathbb{R}^{\{3,4\}}$)

ϕ symplectic $\Longleftrightarrow$ it preserves the areas of all loops.

Continuous Symplectic Topology.

<u>Thm</u> (Eliashberg, Gromov) (1980's) $\{\phi_i\} \subset Symp(M)$

$$\phi_i \xrightarrow[\text{uniformly}]{C^0} \phi_0 \in Diff(M) \implies \phi_0 \text{ is symplectic.}$$

So a C^1 property persists under C^0 convergence!

<u>Def'n</u>: $\phi \in Homeo(M)$ is called "symplectic" if it is the uniform-C^0 limit of a sequence in $Symp(M)$. Set of all: "$Symp_{eo}(M)$".

I.e.

$$Symp_{eo}(M) = \overline{Symp(M)} \subset Homeo(M)$$

<u>Def'n</u> "Lagrangian Submanifolds". The graph

of $\phi \in \mathrm{Diff}(M)$ is Lagrangian iff $\phi \in \mathrm{Symp}(M)$.
(rel $(M \times M, \omega \oplus (-\omega))$)

Thm 2 (Humilière, Leclercq, S) If $L \subset M$ is Lagrangian, $\phi \in \mathrm{Symp}_{eo}(M)$ s.t. $\phi(L)$ is smooth then $\phi(L)$ is Lagrangian.

Thm 2 implies thm 1!

Pf of thm 2 uses "Lagrangian Floer Theory" and "relative capacities".

Thm 3: (Buhovsky-Opshtein) given $\gamma_{1,2}: S^1 \to \mathbb{R}^{2n}$, $n \geq 2$ w/ (possibly) different areas, $\exists\, \phi \in \mathrm{Symp}_{eo}(M)$ s.t. $\phi \circ \gamma_1 = \gamma_2$.

From here on, $\dim(M) = 2$; $(S^2, \omega) = M$

$$\mathcal{H} := \mathrm{Homeo}^{\omega}(S^2) = \mathrm{Symp}_{eo}(S^2)$$

Q (Beguin, Crovisier, LeRoux) $\theta \in \mathcal{H}$, $\mathcal{C}(\theta) = \left\{ \begin{matrix} \psi\theta\psi^{-1}: \\ \psi \in \mathcal{H} \end{matrix} \right\}$ can $\mathcal{C}(\theta)$ be dense in $\mathcal{H}$ (C^0 sense)?

Thm 4 (S.) If $\theta \neq Id$, $\overline{\mathcal{C}(\theta)} \not\ni Id$.

1. Motivation: we don't even know if $\mathcal{H}$ is simple.
2. Can we find $\gamma: \mathcal{H} \to \mathbb{R}^k$ (non-trivial) s.t.
   - i. $\gamma(\psi\theta\psi^{-1}) = \gamma(\theta)$
   - ii. γ is continuous

?

(if there is a dense conjugacy class, γ has to be constant)