Pensieve header: Calculations appearing in the WKO4 paper.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\WKO4"];
```

Section 1 - Introduction

Initialization

```
<< FreeLie.m;
<< AwCalculus.m;
$SeriesShowDegree = 4;
```

Initialization

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, 〈〉, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
  CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

Initialization

```
FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

Initialization

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm,
  hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

Initialization

```
AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

Section 2.2 - Some Preliminaries on Lie Algebras and Cyclic Words

alphabetagamma

```
x = LW@"x"; y = LW@"y";
{α, β, γ} = LS /@ {x + b[x, y], y - b[x, b[x, y]], x + y - 2 b[x, y]}
```

alphabetagamma

$\{LS[\overline{x}, \overline{xy}, 0, 0, \ldots], LS[\overline{y}, 0, -\overline{x\overline{xy}}, 0, \ldots], LS[\overline{x} + \overline{y}, -2\,\overline{xy}, 0, 0, \ldots]\}$

BracketExample

```
{b[α, β], b[α, b[β, γ]] + b[β, b[γ, α]] + b[γ, b[α, β]]}
```

BracketExample

$\{LS[0, \overline{xy}, \overline{\overline{xy}y}, -\overline{x\overline{x\overline{xy}}}, \ldots], LS[0, 0, 0, 0, \ldots]\}$

bch

```
bch = BCH[x, y]
```

bch

$\mathrm{LS}\left[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y},\ \frac{1}{24}\overline{x\overline{\overline{xy}y}},\ \ldots\right]$

```
bch@3
```

$\frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y}$

bch16

```
Timing@{Length@(bch@16), (bch@16)[[1090 ;; 1092]]}
```

bch16

$\left\{25.625,\ \left\{2181,\ -\frac{17\,\overline{x\,x\,x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\,x\,\overline{\overline{xy}y}\,y}}{179\,625\,600}+\frac{389\,\overline{x\,x\,x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\overline{xy}\,x\overline{xy}}}{1\,320\,883\,200}+\frac{53\,\overline{x\,x\,x\overline{xy}\,\overline{xy}\,x\overline{xy}\,x\overline{xy}\,\overline{\overline{xy}y}}}{1\,089\,728\,640}\right\}\right\}$

omegas

```
{ω1, ω2} = CWS /@ {cw[x] - 3 cw[y, x, x], cw[y] + cw[y, y]}
```

omegas

$\{\mathrm{CWS}[\widehat{x},\ 0,\ -3\,\widehat{xxy},\ 0,\ \ldots],\ \mathrm{CWS}[\widehat{y},\ \widehat{yy},\ 0,\ 0,\ \ldots]\}$

DegreeScale

```
DegreeScale[h] /@ {ω1, ω2}
```

DegreeScale

$\left\{\mathrm{CWS}\left[h\,\widehat{x},\ 0,\ -3\,h^3\,\widehat{xxy},\ 0,\ \ldots\right],\ \mathrm{CWS}\left[h\,\widehat{y},\ h^2\,\widehat{yy},\ 0,\ 0,\ \ldots\right]\right\}$

TangentialDerivative

```
{λ = ⟨x → α, y → β⟩, γ // Dλ}
```

TangentialDerivative

$\left\{\left\langle \overline{x}\to \mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \overline{y}\to\mathrm{LS}\left[\overline{y},\ 0,\ -\overline{x\overline{xy}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right]\right\}$

```
α
```

$\mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots]$

```
γ
```

$\mathrm{LS}[\overline{x}+\overline{y},\ -2\,\overline{xy},\ 0,\ 0,\ \ldots]$

tb

```
λ1 = λ; λ2 = ⟨x → β, y → γ⟩; Rol1 = tb[λ1, λ2]
```

tb

$\left\langle \overline{x}\to\mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right],\ \overline{y}\to\mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ \ldots\right]\right\rangle$

```
Rol1x@{8}
```

$\mathrm{LS}\left[0,\ 0,\ \overline{x\overline{xy}},\ -\overline{x\overline{\overline{xy}y}},\ -\overline{\overline{x\overline{xy}}\,\overline{xy}},\ -\overline{x\,x\,x\,\overline{\overline{xy}y}},\ 0,\ 0,\ \ldots\right]$

tb2

$$\texttt{lhs} = D_{tb[\lambda 1,\lambda 2]}[\omega_1];\ \texttt{rhs} = b[D_{\lambda 1}, D_{\lambda 2}][\omega_1];$$
$$\{\texttt{lhs}@\{8\},\ (\texttt{lhs} \equiv \texttt{rhs})@\{8\}\}$$

tb2

$$\{CWS[0, 0, 0, 0, 0, 0, 0, 18\,\overline{xxxxyxyy} - 18\,\overline{xxxxyyxy} - 36\,\overline{xxxyxxyy} + 36\,\overline{xxxyyxxy}, \ldots],\ BS[9\,True, \ldots]\}$$

TestingGammaODE

$$\texttt{lhs} = \partial_t \Gamma_t[\lambda];\ \texttt{rhs} = \lambda\ //\ e^{-t D_\lambda}\ //\ \texttt{adSeries}\left[\frac{ad}{e^{ad}-1}, \Gamma_t[\lambda]\right];$$
$$\{\Gamma_0[\lambda],\ \texttt{lhs},\ (\texttt{lhs} \equiv \texttt{rhs})@\{6\}\}$$

TestingGammaODE

$$\{\langle \overline{x} \to LS[0, 0, 0, 0, \ldots],\ \overline{y} \to LS[0, 0, 0, 0, \ldots]\rangle,$$
$$\Big\langle \overline{x} \to LS\Big[\overline{x}, \overline{x\,y}, -t\,\overline{x\,\overline{x\,y}}, \frac{1}{4} t^2\,\overline{x\,\overline{x\,\overline{x\,y}}} - t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big],$$
$$\overline{y} \to LS\Big[\overline{y}, 0, -\overline{x\,\overline{x\,y}}, -t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big]\Big\rangle,\ BS[7\,True, \ldots]\}$$

TestingGamma

$$\{\gamma\ //\ e^{-t D_\lambda},\ \gamma\ //\ CC[\Gamma_t[\lambda]]\}$$

TestingGamma

$$\{LS[\overline{x} + \overline{y}, -2\,\overline{x\,y}, -t\,\overline{x\,\overline{x\,y}}, t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots],\ LS[\overline{x} + \overline{y}, -2\,\overline{x\,y}, -t\,\overline{x\,\overline{x\,y}}, t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots]\}$$

$$\texttt{Rol2} = (\gamma\ //\ e^{-t D_\lambda}) \equiv (\gamma\ //\ CC[\Gamma_t[\lambda]])$$

BS[5 True, ...]

```
Rol2@{8}
```

BS[9 True, ...]

```
Rol2 // FullForm
```

BooleanSequence[BooleanSequence$7430]

TestingLambdaODE

$$\texttt{lhs} = \partial_t \Lambda_t[\lambda];\ \texttt{rhs} = \lambda\ //\ e^{D_{\Lambda_t[\lambda]}}\ //\ \texttt{adSeries}\left[\frac{ad}{e^{ad}-1}, \Lambda_t[\lambda], tb\right];$$
$$\{\Lambda_0[\lambda],\ \texttt{lhs},\ (\texttt{lhs} \equiv \texttt{rhs})@\{6\}\}$$

TestingLambdaODE

$$\{\langle \overline{x} \to LS[0, 0, 0, 0, \ldots],\ \overline{y} \to LS[0, 0, 0, 0, \ldots]\rangle,$$
$$\Big\langle \overline{x} \to LS\Big[\overline{x}, \overline{x\,y}, t\,\overline{x\,\overline{x\,y}}, \frac{1}{2} t^2\,\overline{x\,\overline{x\,\overline{x\,y}}} + t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big],$$
$$\overline{y} \to LS\Big[\overline{y}, 0, -\overline{x\,\overline{x\,y}}, t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big]\Big\rangle,\ BS[7\,True, \ldots]\}$$

TestingLambda

$$\{\gamma\ //\ CC[t\,\lambda],\ \gamma\ //\ e^{-D_{\Lambda_t[\lambda]}}\}$$

TestingLambda

$$\{LS\Big[\overline{x} + \overline{y}, -2\,\overline{x\,y}, -t\,\overline{x\,\overline{x\,y}}, -\frac{1}{2} t^2\,\overline{x\,\overline{x\,\overline{x\,y}}} + t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big],$$
$$LS\Big[\overline{x} + \overline{y}, -2\,\overline{x\,y}, -t\,\overline{x\,\overline{x\,y}}, -\frac{1}{2} t^2\,\overline{x\,\overline{x\,\overline{x\,y}}} + t\,\overline{x\,\overline{\overline{x\,y}\,y}}, \ldots\Big]\}$$

{(λ // Γ // Λ)@{7}, λ@{7}}

$$\left\{\left\langle\overline{x}\to \mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ 0,\ 0,\ 0,\ \ldots],\ \overline{y}\to \mathrm{LS}\left[\overline{y},\ 0,\ -\overline{x\overline{xy}},\ 0,\ 0,\ 0,\ 0,\ \ldots\right]\right\rangle,\right.$$
$$\left.\left\langle\overline{x}\to \mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ 0,\ 0,\ 0,\ \ldots],\ \overline{y}\to \mathrm{LS}\left[\overline{y},\ 0,\ -\overline{x\overline{xy}},\ 0,\ 0,\ 0,\ 0,\ \ldots\right]\right\rangle\right\}$$

λλ = ⟨x → LS[{x, y}, λλ1s], y → LS[{x, y}, λλ2s]⟩

$$\left\langle\overline{x}\to \mathrm{LS}\left[\overline{x}\,\lambda\lambda1s[x]+\overline{y}\,\lambda\lambda1s[y],\ \overline{xy}\,\lambda\lambda1s[x,y],\ \overline{x\overline{xy}}\,\lambda\lambda1s[x,x,y]+\overline{\overline{xy}y}\,\lambda\lambda1s[x,y,y],\right.\right.$$
$$\left.\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x,x,x,y]+\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x,x,y,y]+\overline{\overline{\overline{xy}y}y}\,\lambda\lambda1s[x,y,y,y],\ \ldots\right],$$
$$\overline{y}\to \mathrm{LS}\left[\overline{x}\,\lambda\lambda2s[x]+\overline{y}\,\lambda\lambda2s[y],\ \overline{xy}\,\lambda\lambda2s[x,y],\ \overline{x\overline{xy}}\,\lambda\lambda2s[x,x,y]+\overline{\overline{xy}y}\,\lambda\lambda2s[x,y,y],\right.$$
$$\left.\left.\overline{x\overline{x\overline{xy}}}\,\lambda\lambda2s[x,x,x,y]+\overline{x\overline{\overline{xy}y}}\,\lambda\lambda2s[x,x,y,y]+\overline{\overline{\overline{xy}y}y}\,\lambda\lambda2s[x,y,y,y],\ \ldots\right]\right\rangle$$

λλ // Γ

$$\left\langle\overline{x}\to \mathrm{LS}\left[\overline{x}\,\lambda\lambda1s[x]+\overline{y}\,\lambda\lambda1s[y],\right.\right.$$
$$-\frac{1}{2}\overline{xy}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]+\overline{xy}\,\lambda\lambda1s[x,y]+\frac{1}{2}\overline{xy}\,\lambda\lambda1s[y]\,\lambda\lambda2s[x],$$
$$\frac{1}{12}\overline{x\overline{xy}}\,\lambda\lambda1s[x]^2\,\lambda\lambda1s[y]+\frac{1}{12}\overline{\overline{xy}y}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]^2-\frac{1}{2}\overline{x\overline{xy}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[x,y]-$$
$$\frac{1}{2}\overline{\overline{xy}y}\,\lambda\lambda1s[y]\,\lambda\lambda1s[x,y]+\overline{x\overline{xy}}\,\lambda\lambda1s[x,x,y]+\overline{\overline{xy}y}\,\lambda\lambda1s[x,y,y]-$$
$$\frac{1}{4}\overline{x\overline{xy}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]\,\lambda\lambda2s[x]-\frac{1}{12}\overline{\overline{xy}y}\,\lambda\lambda1s[y]^2\,\lambda\lambda2s[x]+$$
$$\frac{1}{2}\overline{x\overline{xy}}\,\lambda\lambda1s[x,y]\,\lambda\lambda2s[x]+\frac{1}{6}\overline{x\overline{xy}}\,\lambda\lambda1s[y]\,\lambda\lambda2s[x]^2+\frac{1}{2}\overline{\overline{xy}y}\,\lambda\lambda1s[y]\,\lambda\lambda2s[x,y],$$
$$-\frac{1}{24}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x]^2\,\lambda\lambda1s[y]^2+\frac{1}{12}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x]^2\,\lambda\lambda1s[x,y]+$$
$$\frac{1}{3}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]\,\lambda\lambda1s[x,y]+\frac{1}{12}\overline{\overline{\overline{xy}y}y}\,\lambda\lambda1s[y]^2\,\lambda\lambda1s[x,y]-$$
$$\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x,y]^2-\frac{1}{2}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[x,x,y]-$$
$$\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[y]\,\lambda\lambda1s[x,x,y]-\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[x,y,y]-$$
$$\frac{1}{2}\overline{\overline{\overline{xy}y}y}\,\lambda\lambda1s[y]\,\lambda\lambda1s[x,y,y]+\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x,x,x,y]+\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x,x,y,y]+$$
$$\overline{\overline{\overline{xy}y}y}\,\lambda\lambda1s[x,y,y,y]+\frac{1}{24}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x]^2\,\lambda\lambda1s[y]\,\lambda\lambda2s[x]+$$
$$\frac{1}{12}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]^2\,\lambda\lambda2s[x]-\frac{1}{4}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[x,y]\,\lambda\lambda2s[x]-$$
$$\frac{5}{12}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[y]\,\lambda\lambda1s[x,y]\,\lambda\lambda2s[x]+\frac{1}{2}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x,x,y]\,\lambda\lambda2s[x]+$$
$$\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x,y,y]\,\lambda\lambda2s[x]-\frac{1}{12}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]\,\lambda\lambda2s[x]^2-$$
$$\frac{1}{24}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[y]^2\,\lambda\lambda2s[x]^2+\frac{1}{6}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[x,y]\,\lambda\lambda2s[x]^2+$$
$$\frac{1}{24}\overline{x\overline{x\overline{xy}}}\,\lambda\lambda1s[y]\,\lambda\lambda2s[x]^3-\frac{1}{4}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x]\,\lambda\lambda1s[y]\,\lambda\lambda2s[x,y]-$$
$$\frac{1}{12}\overline{\overline{\overline{xy}y}y}\,\lambda\lambda1s[y]^2\,\lambda\lambda2s[x,y]+\frac{1}{2}\overline{x\overline{\overline{xy}y}}\,\lambda\lambda1s[x,y]\,\lambda\lambda2s[x,y]+$$

$$\frac{1}{3}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[x,y]+\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x,x,y]+$$
$$\frac{1}{2}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x,y,y],\ldots\Big],\ \overline{y}\to\mathrm{LS}\Big[\overline{x}\,\lambda\lambda\mathrm{2s}[x]+\overline{y}\,\lambda\lambda\mathrm{2s}[y],$$
$$-\frac{1}{2}\overline{x\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]+\frac{1}{2}\overline{x\,y}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]+\overline{x\,y}\,\lambda\lambda\mathrm{2s}[x,y],$$
$$\frac{1}{6}\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]^2\,\lambda\lambda\mathrm{2s}[x]-\frac{1}{2}\overline{x\,\overline{x\,y}}\,\lambda\lambda\mathrm{1s}[x,y]\,\lambda\lambda\mathrm{2s}[x]-\frac{1}{12}\overline{x\,\overline{x\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]^2-$$
$$\frac{1}{4}\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]+\frac{1}{12}\overline{x\,\overline{x\,y}}\,\lambda\lambda\mathrm{2s}[x]^2\,\lambda\lambda\mathrm{2s}[y]+$$
$$\frac{1}{12}\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]^2-\frac{1}{2}\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x,y]+\frac{1}{2}\overline{x\,\overline{x\,y}}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[x,y]+$$
$$\frac{1}{2}\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{2s}[y]\,\lambda\lambda\mathrm{2s}[x,y]+\overline{x\,\overline{x\,y}}\,\lambda\lambda\mathrm{2s}[x,x,y]+\overline{\overline{x\,y}\,y}\,\lambda\lambda\mathrm{2s}[x,y,y],$$
$$-\frac{1}{24}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]^3\,\lambda\lambda\mathrm{2s}[x]+\frac{1}{3}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{1s}[x,y]\,\lambda\lambda\mathrm{2s}[x]-$$
$$\frac{1}{2}\overline{x\,\overline{x\,\overline{x\,y}}}\,\lambda\lambda\mathrm{1s}[x,x,y]\,\lambda\lambda\mathrm{2s}[x]-\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[x,y,y]\,\lambda\lambda\mathrm{2s}[x]+$$
$$\frac{1}{24}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]^2\,\lambda\lambda\mathrm{2s}[x]^2-\frac{1}{12}\overline{x\,\overline{x\,\overline{x\,y}}}\,\lambda\lambda\mathrm{1s}[x,y]\,\lambda\lambda\mathrm{2s}[x]^2+$$
$$\frac{1}{12}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]^2\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]-\frac{1}{4}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[x,y]\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]-$$
$$\frac{1}{12}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]^2\,\lambda\lambda\mathrm{2s}[y]-\frac{1}{24}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]^2+$$
$$\frac{1}{24}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[x]^2\,\lambda\lambda\mathrm{2s}[y]^2+\frac{1}{6}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]^2\,\lambda\lambda\mathrm{2s}[x,y]-$$
$$\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[x,y]\,\lambda\lambda\mathrm{2s}[x,y]-\frac{5}{12}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[x,y]+$$
$$\frac{1}{12}\overline{x\,\overline{x\,\overline{x\,y}}}\,\lambda\lambda\mathrm{2s}[x]^2\,\lambda\lambda\mathrm{2s}[x,y]-\frac{1}{4}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[y]\,\lambda\lambda\mathrm{2s}[x,y]+$$
$$\frac{1}{3}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[y]\,\lambda\lambda\mathrm{2s}[x,y]+\frac{1}{12}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{2s}[y]^2\,\lambda\lambda\mathrm{2s}[x,y]+$$
$$\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[x,y]^2-\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x,x,y]+$$
$$\frac{1}{2}\overline{x\,\overline{x\,\overline{x\,y}}}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[x,x,y]+\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[y]\,\lambda\lambda\mathrm{2s}[x,x,y]-$$
$$\frac{1}{2}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{1s}[y]\,\lambda\lambda\mathrm{2s}[x,y,y]+\frac{1}{2}\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[x]\,\lambda\lambda\mathrm{2s}[x,y,y]+$$
$$\frac{1}{2}\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{2s}[y]\,\lambda\lambda\mathrm{2s}[x,y,y]+\overline{x\,\overline{x\,\overline{x\,y}}}\,\lambda\lambda\mathrm{2s}[x,x,x,y]+$$
$$\overline{x\,\overline{\overline{x\,y}\,y}}\,\lambda\lambda\mathrm{2s}[x,x,y,y]+\overline{\overline{\overline{x\,y}\,y}\,y}\,\lambda\lambda\mathrm{2s}[x,y,y,y],\ldots\Big]\Big\rangle$$

Unclassified aside: an alternative formulation of Λ (on March 1, 2015, this took 61 Seconds to degree 8):

```
λ3 = ⟨x → RandomLieSeries[{x, y}], y → RandomLieSeries[{x, y}]⟩;
{lhs = λ3 // EulerE // adSeries[(e^ad - 1)/ad, λ3] // RC[-λ3],
  rhs = Λ[λ3] // EulerE // adSeries[(e^ad - 1)/ad, Λ[λ3], tb]; (lhs ≡ rhs)@{7}} // Timing
```

$\{17.7188,$
$\{\langle \overline{x} \to \mathrm{LS}[-2\,\overline{x} - \overline{y},\ -2\,\overline{xy},\ -\overline{x\overline{xy}} + \frac{15}{2}\overline{\overline{xy}y},\ -\frac{7}{2}\overline{x\overline{x\overline{xy}}} + \frac{59}{6}\overline{x\overline{\overline{xy}y}} - \frac{1}{6}\overline{\overline{\overline{xy}y}y},\ \ldots],\ \overline{y} \to$
$\mathrm{LS}[-2\,\overline{x},\ 0,\ 2\,\overline{x\overline{xy}} + \overline{\overline{xy}y},\ \frac{5}{2}\overline{x\overline{x\overline{xy}}} - \frac{17}{2}\overline{x\overline{\overline{xy}y}} + \frac{2}{3}\overline{\overline{\overline{xy}y}y},\ \ldots]\rangle,\ \mathrm{BS}[8\ \mathrm{True},\ \ldots]\}\}$

CCAndRC

```
{α // CCx[-γ], α // CCx[-γ] // RCx[γ], α // CCx[-γ] // CCx[γ]}
```

CCAndRC

$\{\mathrm{LS}[\overline{x},\ 2\,\overline{xy},\ -\frac{5}{2}\overline{x\overline{xy}} + \frac{3}{2}\overline{\overline{xy}y},\ \frac{7}{6}\overline{x\overline{x\overline{xy}}} - \frac{23}{6}\overline{x\overline{\overline{xy}y}} + \frac{2}{3}\overline{\overline{\overline{xy}y}y},\ \ldots],$
$\mathrm{LS}[\overline{x},\ \overline{xy},\ 0,\ 0,\ \ldots],\ \mathrm{LS}[\overline{x},\ \overline{xy},\ -\overline{x\overline{xy}},\ 2\,\overline{x\overline{x\overline{xy}}} + \overline{x\overline{\overline{xy}y}},\ \ldots]\}$

```
γ
```

$\mathrm{LS}[\overline{x} + \overline{y},\ -2\,\overline{xy},\ 0,\ 0,\ \ldots]$

tru

```
u = LW@"u"; v = LW@"v";
With[{γ = b[b[v, u], u]}, tru[γ]]
```

tru

$-\widehat{uv}$

divu

```
With[{γ = u + b[b[v, u], u]}, divu[γ]]
```

divu

$\widehat{u} - \widehat{uuv}$

Ju

```
Jx[γ]
```

Ju

$\mathrm{CWS}[\widehat{x},\ \frac{5\,\widehat{xy}}{2},\ -\frac{7\,\widehat{xxy}}{6} + \frac{7\,\widehat{xyy}}{6},\ \frac{3\,\widehat{xxxy}}{8} - \frac{11\,\widehat{xxyy}}{4} - \frac{3\,\widehat{xyxy}}{4} + \frac{3\,\widehat{xyyy}}{8},\ \ldots]$

j

```
{div[λ]@{5}, j[λ]@{5}}
```

j

$\{\mathrm{CWS}[\widehat{x} + \widehat{y},\ -\widehat{xy},\ -\widehat{xxy},\ 0,\ 0,\ \ldots],$
$\mathrm{CWS}[\widehat{x} + \widehat{y},\ -\widehat{xy},\ -\widehat{xxy},\ -\widehat{xxyy} + \widehat{xyxy},\ -\widehat{xxxyy} + \widehat{xxyxy},\ \ldots]\}$

cocycle4j

```
lhs = j[BCHtb[λ1, λ2]]; rhs = j[λ1] + e^(Dλ1)[j[λ2]];
{lhs, (lhs ≡ rhs)@{8}}
```

cocycle4j

$\{\mathrm{CWS}[\widehat{x} + 2\,\widehat{y},\ -3\,\widehat{xy},\ 0,\ -9\,\widehat{xxyy} + 9\,\widehat{xyxy},\ \ldots],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\}$

dj

```
ϵ /: ϵ^2 = 0;
{j[ϵ λ], j[ϵ λ] ≡ ϵ div[λ]}
```

dj

```
{CWS[ϵ x + ϵ y, -ϵ xy, -ϵ xxy, 0, ...], BS[5 True, ...]}
```

Section 2.3 - The [AT]-inspired presentation El of $A^w{}_{exp}$

ElSetup

```
x = LW@"x"; y = LW@"y";
{ζa = El[⟨x → LS[x + b[x, y]], y → LS[y - b[x, b[x, y]]]⟩, CWS[cw[x] - 3 cw[x, y, x]]],
 ζb = El[⟨x → LS[y - b[x, y]], y → LS[x + y + b[y, b[x, y]]]⟩, CWS[cw[y] - 2 cw[x, y]]],
 ζc = El[⟨x → LS[x - b[b[x, y], b[x, y]]], y → LS[y + 3 b[x, b[x, y]]]⟩,
   CWS[cw[x] - 2 cw[x, y] + cw[x, y, x]]]}
```

ElSetup

```
{El[⟨x → LS[x, xy, 0, 0, ...], y → LS[y, 0, -xxy, 0, ...]⟩,
  CWS[x, 0, -3 xxy, 0, ...]],
 El[⟨x → LS[y, -xy, 0, 0, ...], y → LS[x + y, 0, -xyy, 0, ...]⟩,
  CWS[y, -2 xy, 0, 0, ...]], El[
  ⟨x → LS[x, 0, 0, 0, ...], y → LS[y, 0, 3 xxy, 0, ...]⟩, CWS[x, -2 xy, xxy, 0, ...]]}
```

ElAssociativity

```
lhs = ζa ** (ζb ** ζc); rhs = (ζa ** ζb) ** ζc;
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

ElAssociativity

```
{El[⟨x → LS[2 x + y, 0, 1/2 xxy, ...], y → LS[x + 3 y, 0, 5/2 xxy - xyy, ...]⟩,
  CWS[2 x + y, -4 xy, -2 xxy, ...]], BS[9 True, ...]}
```

detaExample

```
{ζa // dη^x, ζa // dη^y}
```

detaExample

```
{El[⟨y → LS[y, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨x → LS[x, 0, 0, 0, ...]⟩, CWS[x, 0, 0, 0, ...]]}
```

dA1

```
{ζd = El[λ, CWS[0]], ζd // dA}
```

dA1

```
{El[⟨x → LS[x, xy, 0, 0, ...], y → LS[y, 0, -xxy, 0, ...]⟩, CWS[0, 0, 0, 0, ...]],
 El[⟨x → LS[-x, -xy, 0, 0, ...], y → LS[-y, 0, xxy, 0, ...]⟩,
  CWS[-x - y, xy, xxy, xxyy - xyxy, ...]]}
```

dA2

```
(ζd ≡ (ζd // dA // dA))@{8}
```

dA2

```
BS[9 True, ...]
```

dA3

```
lhs = (ζa ** ζb) // dA; rhs = (ζb // dA) ** (ζa // dA);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dA3

$$\left\{\mathrm{El}\left[\left\langle \bar{x}\to \mathrm{LS}\left[-\bar{x}-\bar{y},\ 0,\ -\frac{1}{2}\,\overline{x\,\overline{x\,y}},\ \ldots\right],\ \bar{y}\to \mathrm{LS}\left[-\bar{x}-2\,\bar{y},\ 0,\ \frac{1}{2}\,\overline{x\,\overline{x\,y}}+\overline{\overline{x\,y}\,y},\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[-\hat{y},\ -2\,\widehat{xy},\ -2\,\widehat{xxy}-\widehat{xyy},\ \ldots\right]\right],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\right\}$$

dS

```
ζd // dS
```

dS

$$\mathrm{El}\left[\left\langle \bar{x}\to \mathrm{LS}\left[\bar{x},\ -\overline{x\,y},\ 0,\ 0,\ \ldots\right],\ \bar{y}\to \mathrm{LS}\left[\bar{y},\ 0,\ -\overline{x\,\overline{x\,y}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\hat{x}+\hat{y},\ \widehat{xy},\ -\widehat{xxy},\ \widehat{xxyy}-\widehat{xyxy},\ \ldots\right]\right]$$

dD1

```
{ζa, ζa // dΔ[y, y, z]}
```

dD1

$$\left\{\mathrm{El}\left[\left\langle \bar{x}\to \mathrm{LS}\left[\bar{x},\ \overline{x\,y},\ 0,\ 0,\ \ldots\right],\ \bar{y}\to \mathrm{LS}\left[\bar{y},\ 0,\ -\overline{x\,\overline{x\,y}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\hat{x},\ 0,\ -3\,\widehat{xxy},\ 0,\ \ldots\right]\right],\ \mathrm{El}\left[\left\langle z\to \mathrm{LS}\left[\bar{y}+\bar{z},\ 0,\ -\overline{x\,\overline{x\,y}}-\overline{x\,\overline{x\,z}},\ 0,\ \ldots\right],\ \bar{x}\to \mathrm{LS}\left[\bar{x},\ \overline{x\,y}+\overline{x\,z},\ 0,\ 0,\ \ldots\right],\ \bar{y}\to \mathrm{LS}\left[\bar{y}+\bar{z},\ 0,\ -\overline{x\,\overline{x\,y}}-\overline{x\,\overline{x\,z}},\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\hat{x},\ 0,\ -3\,\widehat{xxy}-3\,\widehat{xxz},\ 0,\ \ldots\right]\right]\right\}$$

dD2

```
lhs = (ζa ** ζb) // dΔ[y, y, z]; rhs = (ζa // dΔ[y, y, z]) ** (ζb // dΔ[y, y, z]);
{lhs@{3}, (lhs ≡ rhs)@{8}}
```

dD2

$$\left\{\mathrm{El}\left[\left\langle z\to \mathrm{LS}\left[\bar{x}+2\,\bar{y}+2\,\bar{z},\ 0,\ -\frac{1}{2}\,\overline{x\,\overline{x\,y}}-\frac{1}{2}\,\overline{x\,\overline{x\,z}}-\overline{x\,\overline{y\,z}}-\overline{\overline{x\,y}\,y}-2\,\overline{\overline{x\,z}\,y}-\overline{\overline{x\,z}\,z},\ \ldots\right],\ \bar{x}\to \mathrm{LS}\left[\bar{x}+\bar{y}+\bar{z},\ 0,\ \frac{1}{2}\,\overline{x\,\overline{x\,y}}+\frac{1}{2}\,\overline{x\,\overline{x\,z}},\ \ldots\right],\ \bar{y}\to \mathrm{LS}\left[\bar{x}+2\,\bar{y}+2\,\bar{z},\ 0,\ -\frac{1}{2}\,\overline{x\,\overline{x\,y}}-\frac{1}{2}\,\overline{x\,\overline{x\,z}}-\overline{x\,\overline{y\,z}}-\overline{\overline{x\,y}\,y}-2\,\overline{\overline{x\,z}\,y}-\overline{\overline{x\,z}\,z},\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\hat{x}+\hat{y}+\hat{z},\ -2\,\widehat{xy}-2\,\widehat{xz},\ -3\,\widehat{xxy}-3\,\widehat{xxz},\ \ldots\right]\right],\ \mathrm{BS}[9\ \mathrm{True},\ \ldots]\right\}$$

Section 2.4 - The factored presentation Ef of $A^w{}_{exp}$ and its stronger precursor Es

EsSetup1

```
ζa = Es[⟨1 → LS[u + b[u, v]], 2 → LS[v - b[u, b[u, v]]], 3 → LS[u - b[b[u, v], b[u, v]]]⟩,
   CWS[cw[u] - 3 cw[u, v, u]]]
```

EsSetup1

$$\mathrm{Es}\left[\left\langle 1\to \mathrm{LS}\left[\bar{u},\ \overline{u\,v},\ 0,\ 0,\ \ldots\right],\ 2\to \mathrm{LS}\left[\bar{v},\ 0,\ -\overline{u\,\overline{u\,v}},\ 0,\ \ldots\right],\ 3\to \mathrm{LS}\left[\bar{u},\ 0,\ 0,\ 0,\ \ldots\right]\right\rangle,\ \mathrm{CWS}\left[\hat{u},\ 0,\ -3\,\widehat{uuv},\ 0,\ \ldots\right]\right]$$

EsSetup2

```
ξb = RandomEsSeries[0, {1, 2, 3, 4}];
ξb@{2}
```

EsSetup2

$$\text{Es}\Big[\Big\langle 1 \to \text{LS}\Big[-\overline{1} - 2\,\overline{2} + 2\,\overline{3} - 2\,\overline{4},\ 2\,\overline{12} + \frac{\overline{13}}{2} + \overline{14} - \frac{\overline{23}}{2} - \frac{\overline{24}}{2} + 2\,\overline{34},\ \ldots\Big],$$

$$2 \to \text{LS}\Big[2\,\overline{1} - \overline{2} - 2\,\overline{3} + \overline{4},\ 2\,\overline{12} + \frac{3\,\overline{13}}{2} - 2\,\overline{14} - \overline{23} - \overline{24} - \frac{\overline{34}}{2},\ \ldots\Big],$$

$$3 \to \text{LS}\Big[-\overline{1} + \overline{2} + 2\,\overline{4},\ -2\,\overline{12} + 2\,\overline{13} - \overline{14} - \frac{3\,\overline{23}}{2} + 2\,\overline{24} - 2\,\overline{34},\ \ldots\Big],$$

$$4 \to \text{LS}\Big[-2\,\overline{1} + 2\,\overline{2} + 2\,\overline{3} + \overline{4},\ -\frac{\overline{12}}{2} + \frac{3\,\overline{13}}{2} - 2\,\overline{24} + \overline{34},\ \ldots\Big]\Big\rangle,$$

$$\text{CWS}\Big[\overset{\frown}{3} - \overset{\frown}{4},\ \frac{3\,\overset{\frown}{11}}{2} + \frac{3\,\overset{\frown}{12}}{2} - 2\,\overset{\frown}{13} + \overset{\frown}{14} + \overset{\frown}{22} + 2\,\overset{\frown}{23} - \frac{\overset{\frown}{24}}{2} - 2\,\overset{\frown}{33} - \overset{\frown}{34} + \overset{\frown}{44},\ \ldots\Big]\Big]$$

haction

```
lhs = ξa // hm[1, 2, 4] // tha[u, 4];
rhs = ξa // tha[u, 1] // tha[u, 2] // hm[1, 2, 4];
{lhs, (lhs ≡ rhs)@{8}}
```

haction

$$\Big\{\text{Es}\Big[\Big\langle 3 \to \text{LS}\Big[\overline{u},\ -\overline{uv},\ -\overline{u\,\overline{uv}} + \frac{1}{2}\,\overline{\overline{uv}\,v},\ \frac{3}{2}\,\overline{u\,\overline{u\,\overline{uv}}} + \overline{u\,\overline{\overline{uv}\,v}} - \frac{1}{6}\,\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big],$$

$$4 \to \text{LS}\Big[\overline{u} + \overline{v},\ \frac{\overline{uv}}{2},\ -\frac{23}{12}\,\overline{u\,\overline{uv}} - \frac{5}{12}\,\overline{\overline{uv}\,v},\ \overline{u\,\overline{u\,\overline{uv}}} + \frac{13}{24}\,\overline{u\,\overline{\overline{uv}\,v}} + \frac{1}{12}\,\overline{\overline{\overline{uv}\,v}\,v},\ \ldots\Big]\Big\rangle,$$

$$\text{CWS}\Big[2\,\overset{\frown}{u},\ -\overset{\frown}{uv},\ -\frac{3\,\overset{\frown}{uuv}}{2},\ -\frac{\overset{\frown}{uuuv}}{6} + \overset{\frown}{uuvv} - \overset{\frown}{uvuv},\ \ldots\Big]\Big],\ \text{BS}[9\ \text{True},\ \ldots]\Big\}$$

metaassoc

```
lhs = ξb // dm[1, 2, 1] // dm[1, 3, 1]; rhs = ξb // dm[2, 3, 2] // dm[1, 2, 1];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

metaassoc

$$\Big\{\text{Es}\Big[\Big\langle 1 \to \text{LS}\Big[-2\,\overline{1} + \overline{4},\ -\frac{3\,\overline{14}}{2},\ 20\,\overline{1\,\overline{14}} - \frac{19}{3}\,\overline{\overline{14}\,4},\ \ldots\Big],$$

$$4 \to \text{LS}\Big[2\,\overline{1} + \overline{4},\ \overline{14},\ -\frac{31}{2}\,\overline{1\,\overline{14}} - \frac{13}{6}\,\overline{\overline{14}\,4},\ \ldots\Big]\Big\rangle,$$

$$\text{CWS}\Big[3\,\overset{\frown}{1} - \overset{\frown}{4},\ -3\,\overset{\frown}{11} + \frac{\overset{\frown}{14}}{2} + \overset{\frown}{44},\ \frac{71\,\overset{\frown}{111}}{4} + \frac{19\,\overset{\frown}{114}}{4} - \frac{7\,\overset{\frown}{144}}{6} - \frac{2\,\overset{\frown}{444}}{3},\ \ldots\Big]\Big],\ \text{BS}[6\ \text{True},\ \ldots]\Big\}$$

Section 3.1 - Tangle Invariants

Section 3.1.1 - The General Framework

RDefs

```
Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
```

R3

```
lhs = Rl[1, 2] ** Rl[1, 3] ** Rl[2, 3]; rhs = Rl[2, 3] ** Rl[1, 3] ** Rl[1, 2];
{lhs@{3}, (lhs ≡ rhs)@{5}}
```

R3

$$\{\mathrm{El}[\langle 1\to \mathrm{LS}[0,0,0,\ldots],\ 2\to \mathrm{LS}[\overline{1},0,0,\ldots],\ 3\to \mathrm{LS}[\overline{1}+\overline{2},0,0,\ldots]\rangle,\ \mathrm{CWS}[0,0,0,\ldots]],\ \mathrm{BS}[6\ \mathrm{True},\ldots]\}$$

Section 3.1.2 - The Knot 8_{17} and the Borromean Tangle

817

```
t1 = iRs[12, 1] iRs[2, 7] iRs[8, 3] iRs[4, 11] Rs[16, 5] Rs[6, 13] Rs[14, 9] Rs[10, 15];
Do[t1 = t1 // dm[1, k, 1], {k, 2, 16}];
t1@{6}
```

817

$$\mathrm{Es}\left[\langle 1\to \mathrm{LS}[0,0,0,0,0,0,\ldots]\rangle,\ \mathrm{CWS}\left[0,-\widehat{11},0,-\frac{31\,\widehat{1111}}{12},0,-\frac{1351\,\widehat{111111}}{360},\ldots\right]\right]$$

Borromean

```
t2 = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
(Do[t2 = t2 // dm[r, k, r], {k, 1, 3}]; Do[t2 = t2 // dm[g, k, g], {k, 4, 6}];
 Do[t2 = t2 // dm[b, k, b], {k, 7, 9}]; t2)
```

Borromean

$$\begin{aligned}\mathrm{Es}\Big[\Big\langle b\to \mathrm{LS}\Big[&0,\ \overline{gr},\ \tfrac12\overline{g\overline{gr}}+\overline{\overline{br}g}+\tfrac12\overline{\overline{gr}r},\\ &-\tfrac12\overline{b\overline{\overline{br}g}}+\tfrac16\overline{g\overline{g\overline{gr}}}+\tfrac14\overline{g\overline{\overline{gr}r}}-\tfrac12\overline{\overline{bg}\,\overline{br}}-\tfrac12\overline{\overline{\overline{br}g}g}-\tfrac12\overline{\overline{\overline{br}r}g}+\tfrac16\overline{\overline{\overline{gr}r}r},\ \ldots\Big],\ g\to\\ \mathrm{LS}\Big[&0,\ -\overline{br},\ \tfrac12\overline{b\overline{br}}-\overline{b\overline{gr}}-\overline{\overline{br}g}+\tfrac12\overline{\overline{br}r},\ -\tfrac16\overline{b\overline{b\overline{br}}}-\tfrac12\overline{b\overline{b\overline{gr}}}-\tfrac12\overline{b\overline{g\overline{gr}}}-\tfrac12\overline{b\overline{\overline{br}g}}-\\ &\tfrac14\overline{b\overline{\overline{br}r}}+\tfrac12\overline{b\overline{\overline{gr}r}}+\tfrac12\overline{\overline{bg}\,\overline{br}}+\overline{\overline{br}\,\overline{gr}}-\overline{\overline{b\overline{gr}}g}-\tfrac12\overline{\overline{\overline{br}g}g}+\tfrac12\overline{\overline{\overline{br}r}g}-\tfrac16\overline{\overline{\overline{br}r}r},\ \ldots\Big],\\ r\to \mathrm{LS}\Big[&0,\ \overline{bg},\ \tfrac12\overline{b\overline{bg}}+\overline{b\overline{gr}}+\tfrac12\overline{\overline{bg}g},\ \tfrac16\overline{b\overline{b\overline{bg}}}+\tfrac12\overline{b\overline{b\overline{gr}}}+\\ &\tfrac12\overline{b\overline{g\overline{gr}}}+\tfrac14\overline{b\overline{\overline{bg}g}}+\tfrac12\overline{b\overline{\overline{gr}r}}+\tfrac16\overline{\overline{\overline{bg}g}g},\ \ldots\Big]\Big\rangle,\\ \mathrm{CWS}\Big[&0,0,2\,\widehat{bgr},\ \widehat{bbgr}-\widehat{bgbr}+\widehat{bggr}-\widehat{bgrg}+\widehat{bgrr}-\widehat{brgr},\ \ldots\Big]\Big]\end{aligned}$$

Section 3.2 - Solutions of the Kashiwara-Vergne Equations

Continues pensieve://2013-10/SolvingWKO.nb.

VSetup

```
x = LW["x"]; y = LW["y"]; z = LW["z"];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V0 = Es[⟨x → α, y → β⟩, γ];
```

CapSetup

```
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
```

VCapEqns

```
R4Eqn = V0 ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V0 ;
UnitarityEqn = V0 ** (V0 // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn = (V0 ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);
```

VCapSolution

```
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ}, (ħ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V0@{4}, κ@{6}}
```

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.

VCapSolution

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

VCapSolution

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

VCapSolution

$$\Big\{\mathrm{Es}\Big[\Big\langle \overline{x} \to \mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{7\,\overline{x\,\overline{\overline{xy}\,y}}}{5760} + \frac{\overline{\overline{\overline{xy}\,y}\,y}}{1440},\ \ldots\Big],$$

$$\overline{y} \to \mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\,\overline{x\,\overline{xy}}}}{5760} - \frac{1}{720}\,\overline{x\,\overline{\overline{xy}\,y}} + \frac{1}{720}\,\overline{\overline{\overline{xy}\,y}\,y},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880} + \frac{\overline{xxyy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880},\ \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11\,520},\ 0,\ -\frac{\overline{xxxxxx}}{725\,760},\ \ldots\Big]\Big\}$$

Sinh

```
Series[1/4 Log[(ħ/2)/Sinh[ħ/2]], {ħ, 0, 12}]
```

Sinh

$$-\frac{\hbar^2}{96} + \frac{\hbar^4}{11\,520} - \frac{\hbar^6}{725\,760} + \frac{\hbar^8}{38\,707\,200} - \frac{\hbar^{10}}{1\,916\,006\,400} + \frac{691\,\hbar^{12}}{62\,768\,369\,664\,000} + O[\hbar]^{13}$$

LambdaV

```
Λ[V0]
```

LambdaV

$$\mathrm{El}\Big[\Big\langle \overline{x} \to \mathrm{LS}\Big[0, -\frac{\overline{xy}}{24}, \frac{1}{96}\overline{x\overline{xy}}, \frac{\overline{x\overline{x\overline{xy}}}}{2880} - \frac{1}{480}\overline{x\overline{\overline{xy}y}} + \frac{\overline{\overline{\overline{xy}y}y}}{1440}, \ldots\Big],$$
$$\overline{y} \to \mathrm{LS}\Big[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, \frac{1}{96}\overline{x\overline{xy}}, \frac{1}{960}\overline{x\overline{x\overline{xy}}} - \frac{1}{320}\overline{x\overline{\overline{xy}y}} + \frac{1}{720}\overline{\overline{\overline{xy}y}y}, \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0, -\frac{\overline{xy}}{48}, 0, \frac{\overline{xxxy}}{2880} + \frac{\overline{xxyy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880}, \ldots\Big]\Big]$$

logF

```
logF = Λ[V0]〚1〛 // dσ[{x, y} → {y, x}]
```

logF

$$\Big\langle \overline{x} \to \mathrm{LS}\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \frac{1}{96}\overline{\overline{xy}y}, -\frac{1}{720}\overline{x\overline{x\overline{xy}}} + \frac{1}{320}\overline{x\overline{\overline{xy}y}} - \frac{1}{960}\overline{\overline{\overline{xy}y}y}, \ldots\Big],$$
$$\overline{y} \to \mathrm{LS}\Big[0, \frac{\overline{xy}}{24}, \frac{1}{96}\overline{\overline{xy}y}, -\frac{\overline{x\overline{x\overline{xy}}}}{1440} + \frac{1}{480}\overline{x\overline{\overline{xy}y}} - \frac{\overline{\overline{\overline{xy}y}y}}{2880}, \ldots\Big]\Big\rangle$$

atkv

```
atkv = logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb];
{f = atkv_x, g = atkv_y}
```

atkv

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots\Big],$$
$$\mathrm{LS}\Big[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots\Big]\Big\}$$

On March 1, 2015, the following took 379 seconds in degree 8:

KVTest

$$\Big(\hbar^{-1}\,(\mathrm{LS}[x+y] - \mathrm{BCH}[y, x] \equiv f - g - \mathrm{Ad}[-x][f] + \mathrm{Ad}[y][g]) \bigwedge$$
$$\mathrm{div}_x[f] + \mathrm{div}_y[g] \equiv \frac{1}{2}\,\mathrm{tr}_u\Big[\mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1}, x\Big][u] + \mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1}, y\Big][u] -$$
$$\mathrm{adSeries}\Big[\frac{ad}{e^{ad}-1}, \mathrm{BCH}[x, y]\Big][u]\Big]\Big) @\{6\} \;//\; \mathrm{Timing}$$

KVTest

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

KVTest

```
{16.9688, BS[7 True, ...]}
```

KVDirect

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ħ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡ 1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] +
       adSeries[ad/(e^ad - 1), y][u] - adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F,
 G}
```

KVDirect

$\{LS[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{180}\overline{x\overline{x\overline{xy}}} + \frac{1}{80}\overline{x\overline{\overline{xy}y}} + \frac{1}{360}\overline{\overline{\overline{xy}y}y}, \ldots],$
$LS[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{\overline{xy}y}, -\frac{1}{360}\overline{x\overline{x\overline{xy}}} + \frac{1}{120}\overline{x\overline{\overline{xy}y}} + \frac{1}{180}\overline{\overline{\overline{xy}y}y}, \ldots]\}$

Section 3.3 - The involution τ and the twist equation

Theta

```
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
{Θl[x, y, 1], Θs[x, y, 1]}
```

Theta

$\{El[\langle \overline{x} \to LS[\overline{y}, 0, 0, 0, \ldots], \overline{y} \to LS[\overline{x}, 0, 0, 0, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]],$
$Es[\langle \overline{x} \to LS[\overline{y}, \frac{\overline{xy}}{2}, \frac{1}{6}\overline{x\overline{xy}} - \frac{1}{12}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{x\overline{xy}}} - \frac{1}{24}\overline{x\overline{\overline{xy}y}}, \ldots], \overline{y} \to$
$LS[\overline{x}, -\frac{\overline{xy}}{2}, -\frac{1}{12}\overline{x\overline{xy}} + \frac{1}{6}\overline{\overline{xy}y}, \frac{1}{24}\overline{x\overline{\overline{xy}y}} - \frac{1}{24}\overline{\overline{\overline{xy}y}y}, \ldots]\rangle, CWS[0, 0, 0, 0, \ldots]]\}$

Vtau

```
τV = Rs[x, y] ** (V₀ // dσ[{x, y} → {y, x}]) ** Θs[x, y, -1/2];
(V₀ ≡ τV) @ {6}
```

Vtau

BS[7 True, ...]

Linearized

```
{A = LS[{x, y}, As], B = LS[{x, y}, Bs]};
msgs = SeriesSolve[{A, B},
   ħ^-1 (b[x, A] + b[y, B] ≡ LS[0]) ⋀ (div_x[A] + div_y[B] ≡ CWS[0])];
{A, B}
```

Linearized

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {As[y] → 0}.

Linearized

{LS[0, 0, 0, 0, ...], LS[0, 0, 0, 0, ...]}

msgs
```
Read[msgs]
```
msgs
```
{{ArbitrarilySetting, 1, {Hold[As[y]] → 0}}, {ArbitrarilySetting, 2, {}},
 {ArbitrarilySetting, 3, {}}, {ArbitrarilySetting, 4, {}}}
```
dims
```
A@12; Length[Last[#]] & /@ Read[msgs]
```
dims

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {As[x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y] → 0}.

dims

SeriesSolve::ArbitrarilySetting : In degree 11 arbitrarily setting {As[x, x, x, x, x, x, y, x, y, y, y] → 0}.

dims

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims
```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```
dims1
```
{A1 = LS[{x, y}, A1s], B1 = LS[{x, y}, B1s]};
msgs1 = SeriesSolve[{A1, B1},
   ℏ^-1 (b[x, A1] + b[y, B1] ≡ LS[0]) ⋀
    (div_x[A1] + div_y[B1] ≡ CWS[0]) ⋀ (A1 ≡ (B1 // LieMorphism[x → y, y → x]))];
A1@12; Length[Last[#]] & /@ Read[msgs1]
```
dims1

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {A1s[y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {A1s[x, x, x, x, y, x, y, y] → 0}.

dims1

SeriesSolve::ArbitrarilySetting : In degree 10 arbitrarily setting {A1s[x, x, x, x, x, x, y, x, y, y] → 0}.

dims1

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

dims1
```
{1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 2}
```

Section 3.4 - Drinfel'd Associators

4T
```
{b[t[1, 3], t[4, 2]], b[t[1, 2] + t[1, 3], t[2, 3]]}
```
4T
```
{0, 0}
```
DKExample
```
b[t[1, 3], t[1, 2]]
```
DKExample

$\mathrm{DK}\left[3, -\overline{1\,2}\right]$

DKSExample

```
b[t[1, 3], t[1, 2]] // DKS
```

DKSExample

$$\text{DKS}\left[0,\ -\overline{t_{13}\,t_{23}},\ 0,\ 0,\ \ldots\right]$$

sigmaExample

```
{t[2, 3]^σ[{2,4},{1,5},{3,7,8},{9}] // DKS, t[2, 3]^σ[24,15,378,9] // DKS}
```

sigmaExample

$$\left\{\text{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right],\ \text{DKS}\left[\overline{t_{13}}+\overline{t_{17}}+\overline{t_{18}}+\overline{t_{35}}+\overline{t_{57}}+\overline{t_{58}},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$$

BCH4DK

```
R = DKS[t[1, 2] / 2];
{R ** R^σ[2,3], R ** R^σ[12,3]}
```

BCH4DK

$$\left\{\text{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{23}}}{2},\ -\frac{1}{8}\overline{t_{13}\,t_{23}},\ -\frac{1}{48}\overline{\overline{t_{13}\,t_{23}}\,t_{23}}+\frac{1}{96}\overline{t_{13}\,\overline{t_{13}\,t_{23}}},\ -\frac{1}{384}\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}+\frac{1}{384}\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}},\ \ldots\right],\ \text{DKS}\left[\frac{\overline{t_{12}}}{2}+\frac{\overline{t_{13}}}{2}+\frac{\overline{t_{23}}}{2},\ 0,\ 0,\ 0,\ \ldots\right]\right\}$$

Phi

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1/24; Φ0 = DKS[3, Φs];
SeriesSolve[Φ0,
   (Φ0^σ[3,2,1] ≡ -Φ0) ⋀ (Φ0 ** Φ0^σ[1,23,4] ** Φ0^σ[2,3,4] ≡ Φ0^σ[12,3,4] ** Φ0^σ[1,2,34])];
Φ0@
  {6}
```

Phi

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

Phi

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

Phi

$$\text{DKS}\Big[0,\ \frac{1}{24}\overline{t_{13}\,t_{23}},\ 0,\ -\frac{7\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}{5760}+\frac{7\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}{5760}-\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}{1440},$$
$$0,\ \frac{31\,\overline{\overline{\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}}{967\,680}-\frac{157\,\overline{t_{13}\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,\overline{t_{13}\,t_{23}}}}}{1\,935\,360}-\frac{31\,\overline{\overline{\overline{t_{13}\,t_{23}}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}\,t_{23}}}{387\,072}-$$
$$\frac{31\,\overline{t_{13}\,\overline{\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}\,t_{23}}}}{483\,840}+\frac{11\,\overline{t_{13}\,\overline{\overline{t_{13}\,\overline{t_{13}\,t_{23}}}\,\overline{t_{13}\,t_{23}}}}}{290\,304}+\frac{31\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}}{725\,760}+$$
$$\frac{83\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{\overline{t_{13}\,t_{23}}\,t_{23}}\,t_{23}}}}}{967\,680}-\frac{13\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{\overline{t_{13}\,t_{23}}\,t_{23}}}}}}{241\,920}+\frac{\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,\overline{t_{13}\,t_{23}}}}}}}{60\,480},\ \ldots\Big]$$

Hexagons

```
R = DKS[t[1, 2] / 2];
(R^σ[12,3] ≡ Φ0 ** R^σ[2,3] ** (-Φ0)^σ[1,3,2] ** R^σ[1,3] ** Φ0^σ[3,1,2] ⋀
   (-R)^σ[12,3] ≡ Φ0 ** (-R)^σ[2,3] ** (-Φ0)^σ[1,3,2] ** (-R)^σ[1,3] ** Φ0^σ[3,1,2]) @ {6}
```

Hexagons

```
BS[7 True, ...]
```

Section 3.5 - Associators in $\mathcal{A}^w$

PhiV

```
V12 = V0 // dσ[{x, y} → {1, 2}];
ΦV = (V12 // dA)^σ[12,3] ** (V12 // dA)^σ[1,2] ** V12^σ[2,3] ** V12^σ[1,23]
```

PhiV

Es[

$$\Big\langle 1 \to \mathrm{LS}\Big[0,\ \frac{\overline{2\,3}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1440}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{5760}-\frac{7\,\overline{2\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{2\,\overline{\overline{2\,3}\,3}}}{5760}+\frac{1}{480}\overline{\overline{1\,2}\,\overline{1\,3}}-\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1920}+\frac{1}{640}\overline{\overline{1\,\overline{2\,3}}\,2}-\frac{\overline{\overline{\overline{1\,3}\,2}\,2}}{1152}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1152}-\frac{\overline{\overline{\overline{2\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$2 \to \mathrm{LS}\Big[0,\ -\frac{\overline{1\,3}}{24},\ 0,\ \frac{\overline{1\,\overline{1\,\overline{1\,3}}}}{1440}-\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1152}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{1920}-\frac{1}{480}\overline{1\,\overline{\overline{1\,3}\,2}}-\frac{\overline{1\,\overline{\overline{1\,3}\,3}}}{5760}+\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1152}+\frac{7\,\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{19\,\overline{\overline{1\,3}\,\overline{2\,3}}}{5760}+\frac{7\,\overline{\overline{1\,\overline{2\,3}}\,2}}{1920}+\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}+\frac{7\,\overline{\overline{\overline{1\,3}\,3}\,2}}{5760}+\frac{\overline{\overline{\overline{1\,3}\,3}\,3}}{1440},\ \ldots\Big],$$

$$3 \to \mathrm{LS}\Big[0,\ \frac{\overline{1\,2}}{24},\ 0,\ -\frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{1440}+\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760}+\frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760}-\frac{\overline{1\,\overline{\overline{1\,3}\,2}}}{1440}-\frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1440}+\frac{\overline{\overline{1\,2}\,\overline{1\,3}}}{5760}+\frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1440}-\frac{\overline{\overline{1\,\overline{2\,3}}\,2}}{1152}-\frac{7\,\overline{\overline{\overline{1\,2}\,2}\,2}}{5760}-\frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760}-\frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1440},\ \ldots\Big]\Big\rangle,\ \mathrm{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

PentPhiV

```
ΦV ** ΦV^σ[1,23,4] ** ΦV^σ[2,3,4] ≡ ΦV^σ[12,3,4] ** ΦV^σ[1,2,34]
```

PentPhiV

```
BS[5 True, ...]
```

Phi_is_sder

```
ϕ = (ΦV // Λ)[[1]];
(b[LW@1, ϕ1] + b[LW@2, ϕ2] + b[LW@3, ϕ3])@{6}
```

Phi_is_sder

```
LS[0, 0, 0, 0, 0, 0, ...]
```

DK2Es

```
DK2Es[s___][ζ_] := El[ζ // αMap[s], CWS[0]] // Γ;
DK2Es[1, 2, 3][Φ0]
```

DK2Es

Es[

$$\Big\langle 1 \to \mathrm{LS}\Big[0, \frac{\overline{2\,3}}{24}, 0, -\frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1440} + \frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760} + \frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{5760} - \frac{7\,\overline{2\,\overline{2\,\overline{2\,3}}}}{5760} + \frac{7\,\overline{2\,\overline{\overline{2\,3}\,3}}}{5760} + \frac{1}{480}\,\overline{\overline{1\,2}\,\overline{1\,3}} - \frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1920} + \frac{1}{640}\,\overline{\overline{1\,\overline{2\,3}}\,2} - \frac{\overline{\overline{\overline{1\,3}\,2}\,2}}{1152} - \frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1152} - \frac{\overline{\overline{\overline{2\,3}\,3}\,3}}{1440}, \ldots\Big],$$

$$2 \to \mathrm{LS}\Big[0, -\frac{\overline{1\,3}}{24}, 0, \frac{\overline{1\,\overline{1\,\overline{1\,3}}}}{1440} - \frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{1152} + \frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{1920} - \frac{1}{480}\,\overline{1\,\overline{\overline{1\,3}\,2}} - \frac{\overline{1\,\overline{\overline{1\,3}\,3}}}{5760} + \frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1152} + \frac{7\,\overline{\overline{1\,2}\,\overline{1\,3}}}{5760} + \frac{19\,\overline{\overline{1\,3}\,\overline{2\,3}}}{5760} + \frac{7\,\overline{\overline{1\,\overline{2\,3}}\,2}}{1920} + \frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760} + \frac{7\,\overline{\overline{\overline{1\,3}\,3}\,2}}{5760} + \frac{\overline{\overline{\overline{1\,3}\,3}\,3}}{1440}, \ldots\Big],$$

$$3 \to \mathrm{LS}\Big[0, \frac{\overline{1\,2}}{24}, 0, -\frac{\overline{1\,\overline{1\,\overline{1\,2}}}}{1440} + \frac{\overline{1\,\overline{1\,\overline{2\,3}}}}{5760} + \frac{7\,\overline{1\,\overline{2\,\overline{2\,3}}}}{5760} + \frac{7\,\overline{1\,\overline{\overline{1\,2}\,2}}}{5760} - \frac{\overline{1\,\overline{\overline{1\,3}\,2}}}{1440} - \frac{\overline{1\,\overline{\overline{2\,3}\,3}}}{1440} + \frac{\overline{\overline{1\,2}\,\overline{1\,3}}}{5760} + \frac{\overline{\overline{1\,3}\,\overline{2\,3}}}{1440} - \frac{\overline{\overline{1\,\overline{2\,3}}\,2}}{1152} - \frac{7\,\overline{\overline{\overline{1\,2}\,2}\,2}}{5760} - \frac{7\,\overline{\overline{\overline{1\,3}\,2}\,2}}{5760} - \frac{\overline{\overline{\overline{1\,3}\,3}\,2}}{1440}, \ldots\Big]\Big\rangle, \mathrm{CWS}[0, 0, 0, 0, \ldots]\Big]$$

The computation below takes a a couple of hours and yields "BS[8 True,False,...]":

```
TrueQ[DK2Es[1, 2, 3][Φ0] ≡ ΦV]@{8}
```

BS[8 True, False, ...]

Section 3.6 - Solving the Kashiwara-Vergne Equations Using a Drinfel'd Associator

ZB

```
R = DKS[t[1, 2] / 2];
ZB = (-Φ0)^σ[13,2,4] ** Φ0^σ[1,3,2] ** R^σ[2,3] ** (-Φ0)^σ[1,2,3] ** Φ0^σ[12,3,4]
```

ZB

$$\text{DKS}\Big[\frac{\overline{t_{23}}}{2},\ -\frac{1}{12}\overline{t_{13}t_{23}}-\frac{1}{24}\overline{t_{14}t_{24}}+\frac{1}{24}\overline{t_{14}t_{34}}+\frac{1}{12}\overline{t_{24}t_{34}},\ 0,$$
$$\frac{\overline{\overline{\overline{t_{13}t_{23}}t_{23}}t_{23}}}{5760}+\frac{7\,\overline{\overline{\overline{t_{14}t_{24}}t_{24}}t_{24}}}{5760}+\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{24}}t_{24}}}{1920}-\frac{\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{24}}}{1920}-\frac{7\,\overline{\overline{\overline{t_{14}t_{34}}t_{34}}t_{34}}}{5760}-$$
$$\frac{\overline{\overline{\overline{t_{24}t_{34}}t_{34}}t_{34}}}{5760}+\frac{\overline{\overline{t_{14}\overline{t_{24}t_{34}}}t_{24}}}{1920}+\frac{\overline{\overline{t_{14}t_{24}}\,\overline{t_{14}t_{34}}}}{1920}-\frac{\overline{\overline{t_{14}t_{34}}\,\overline{t_{24}t_{34}}}}{1920}-\frac{1}{720}\overline{t_{13}\overline{\overline{t_{13}t_{23}}t_{23}}}+$$
$$\frac{1}{720}\overline{t_{13}\overline{t_{13}\overline{t_{13}t_{23}}}}-\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{24}}t_{24}}}}{5760}+\frac{7\,\overline{t_{14}\overline{\overline{t_{14}t_{34}}t_{34}}}}{5760}-\frac{\overline{t_{14}\overline{\overline{t_{24}t_{34}}t_{34}}}}{5760}+\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{24}}}}}{1440}-$$
$$\frac{\overline{t_{14}\overline{t_{14}\overline{t_{14}t_{34}}}}}{1440}-\frac{1}{960}\overline{t_{14}\overline{t_{14}\overline{t_{24}t_{34}}}}+\frac{\overline{t_{14}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760}-\frac{1}{960}\overline{t_{24}\overline{\overline{t_{24}t_{34}}t_{34}}}-\frac{\overline{t_{24}\overline{t_{24}\overline{t_{24}t_{34}}}}}{5760},\ \ldots\Big]$$

VfromPhi

```
ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3
```

VfromPhi

$$\text{Es}\Big[\Big\langle 1\to\text{LS}\Big[0,\ -\frac{\overline{2\,4}}{24},\ 0,\ \frac{7\,\overline{2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{7\,\overline{2\,\overline{\overline{2\,4}\,4}}}{5760}+\frac{\overline{\overline{\overline{2\,4}\,4}\,4}}{1440},\ \ldots\Big],$$
$$2\to\text{LS}[0,\ 0,\ 0,\ 0,\ \ldots],\ 3\to\text{LS}\Big[\frac{\overline{2}}{2},\ -\frac{\overline{2\,4}}{12},\ 0,\ \frac{\overline{2\,\overline{2\,\overline{2\,4}}}}{5760}-\frac{1}{720}\overline{2\,\overline{\overline{2\,4}\,4}}+\frac{1}{720}\overline{\overline{\overline{2\,4}\,4}\,4},\ \ldots\Big],$$
$$4\to\text{LS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big\rangle,\ \text{CWS}[0,\ 0,\ 0,\ 0,\ \ldots]\Big]$$

The computation below takes a few hours and yields "BS[8 True,False,...]":

```
VB = ZB // DK2Es[1, 2, 3, 4] // tη^1 // tη^3 // hη^2 // hη^4 // hσ[{1, 3} → {x, y}] //
  tσ[{2, 4} → {x, y}]; TrueQ[VB[[1]] ≡ V0[[1]]] @ {8}
```

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {αs[x, x, x, x, x, y, y] → 0}.

General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

```
BS[8 True, False, ...]
```

nu

```
νinv = Φ0 // DK2Es[1, 2, 3] // dS[2] // dm[3, 2, 2] // dm[2, 1, x]
```

nu

```
Es[⟨x̄ → LS[0, 0, 0, 0, ...]⟩, CWS[0, xx/24, 0, -xxxx/2880, ...]]
```

nucap4

```
(νinv ** Cap ** Cap ** Cap ** Cap)@{6}
```

nucap4

```
Es[⟨x̄ → LS[0, 0, 0, 0, 0, 0, ...]⟩, CWS[0, 0, 0, 0, 0, 0, ...]]
```

Section 3.7 - A Potential S_4 Action on Solutions of KV

rho2

```
ρ2[V_] := V // (-1)^deg;
V1 = Es[⟨x → LS[0], y → LS[-x/2]⟩, CWS[0]] ** V0;
{(V1 ≡ ρ2[V1])@{8}, (V0 ≡ Rs[x, y] ** ρ2[V0])@{8}}
```

rho2

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting {αs[x, x, x, x, y, x, y, y] → 0}.

rho2

```
{BS[9 True, ...], BS[9 True, ...]}
```

rho3

```
ρ3[ξ_Es] := ξ // dS[y] // dΔ[y, y, z] // dm[x, z, x] // dσ[{x, y} → {y, x}];
ξc = RandomEsSeries[1, {x, y}];
ξc ≡ (ξc // ρ3 // ρ3 // ρ3)
```

rho3

```
BS[5 True, ...]
```

V2

```
V2 = V0 ** Θs[x, y, -1/4] **
    Es[⟨x → LS@0, y → LS@0⟩, CWS[cw[x]/12 - cw[y]/12] - (2 Cap⟦2⟧ // tΔ[x, x, y])];
(V2 ≡ ρ3[V2])@{6}
```

V2

```
BS[7 True, ...]
```