Proof of Cayley-Hamilton

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<u>Thm</u> Let R be a commutative ring, $A \in M_{n \times n}(R)$, $\chi_A(t) := \det(tI - A)$. Then $\chi_A(A) = 0$.

<u>Proof</u> Over any commutative ring, any matrix M has an "adjoint matrix" $\operatorname{adj}(M)$, defined using M's minors, and which satisfies $(\operatorname{adj} M) \cdot M = \det(M) \cdot I$. In particular, using the ring $R[t]$ and $M = tI - A$, we have the following equality in $M_{n \times n}(R[t])$:

$$\chi_A(t) \cdot I = \det(tI - A) I = \operatorname{adj}(tI - A) \cdot (tI - A)$$

Let's re-interpret this equality in the ring $M_{n \times n}(R)[t]$, which is isomorphic to $M_{n \times n}(R[t])$. Write $\operatorname{adj}(tI - A) = \sum B_i t^i$, and then

$$\chi_A(t) \cdot I = \left(\sum B_i t^i\right)(tI - A) = \sum B_i t^{i+1} - \sum B_i A t^i$$

Now note that there is a well-defined linear (though not multiplicative) evaluation at A map $ev_A : M_{n \times n}(R)[t] \longrightarrow M_{n \times n}(R)$.

Apply it to both sides of the above equality and get $\chi_A(A) \cdot I = \sum B_i A^{i+1} - \sum B_i A A^i = 0$. $\square$