

Pensieve header: Trying to understand the $U(1)$ -indeterminacy of KV solutions.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2015-12"];
```

```
Get["../Projects/WKO4/FreeLie.m"];
```

```
Get["../Projects/WKO4/AwCalculus.m"];
```

FreeLie` implements / extends

```
{*, +, **, $SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ħ, ↦, ↪}.
```

FreeLie` is in the public domain. Dror Bar-Natan is committed to

support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm,

hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Δ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed

to support it within reason until July 15, 2022. This is version 150909.

```

l = LW["l"]; r = LW["r"]; x = LW["x"]; y = LW["y"]; z = LW["z"];
A = LS[{l, r}, as];
V = El[
  A // LieMorphism[
    {l → ⟨x → LS[y], y → LS[0]⟩, r → ⟨x → LS[0], y → LS[x]⟩}, ⟨x, y⟩, tb],
  CWS[0]
] // Γ;
V@{4}
Es[⟨x̄ → LS[as[l] ȳ, 1/2 as[l] as[r] x̄ȳ + as[l, r] x̄ȳ, 1/6 as[l] as[r]^2 x̄x̄ȳ +
  1/2 as[r] as[l, r] x̄x̄ȳ + as[l, r, r] x̄x̄ȳ - 1/12 as[l]^2 as[r] x̄x̄ȳ + as[l, l, r] x̄x̄ȳ,
  1/24 as[l] as[r]^3 x̄x̄x̄ȳ + 1/6 as[r]^2 as[l, r] x̄x̄x̄ȳ + 1/2 as[r] as[l, r, r] x̄x̄x̄ȳ +
  as[l, r, r, r] x̄x̄x̄ȳ - 1/24 as[l]^2 as[r]^2 x̄x̄x̄ȳ - 1/12 as[l] as[r] as[l, r] x̄x̄x̄ȳ +
  1/2 as[r] as[l, l, r] x̄x̄x̄ȳ + as[l, l, r, r] x̄x̄x̄ȳ + as[l, l, l, r] x̄x̄x̄ȳ, ...⟩,
  ȳ → LS[as[r] x̄, -1/2 as[l] as[r] x̄ȳ + as[l, r] x̄ȳ, -1/12 as[l] as[r]^2 x̄x̄ȳ +
  as[l, r, r] x̄x̄ȳ + 1/6 as[l]^2 as[r] x̄x̄ȳ - 1/2 as[l] as[l, r] x̄x̄ȳ + as[l, l, r] x̄x̄ȳ,
  as[l, r, r, r] x̄x̄x̄ȳ + 1/24 as[l]^2 as[r]^2 x̄x̄x̄ȳ - 1/12 as[l] as[r] as[l, r] x̄x̄x̄ȳ -
  1/2 as[l] as[l, r, r] x̄x̄x̄ȳ + as[l, l, r, r] x̄x̄x̄ȳ - 1/24 as[l]^3 as[r] x̄x̄x̄ȳ +
  1/6 as[l]^2 as[l, r] x̄x̄x̄ȳ - 1/2 as[l] as[l, l, r] x̄x̄x̄ȳ +
  as[l, l, l, r] x̄x̄x̄ȳ, ...⟩], CWS[0, 0, 0, 0, ...]]

Rl[a_, b_] := El[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRl[a_, b_] := El[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
Θl[x_, y_, s_] := El[⟨x → LS[s LW@y], y → LS[s LW@x]⟩, CWS[0]];
Θs[x_, y_, s_] := Θl[x, y, s] // Γ;
R4Eqn[V_] := V** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn[V_] := V** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]];
CapEqn[V_, Cap_] := (V** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
  (Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]);

```

```
as["1"] = -1/4;
msgS = SeriesSolve[A, ħ-1 R4Eqn[V]];
```

A[1]

$$-\frac{\overline{1}}{4} + \frac{\overline{r}}{4}$$

A[1]

$$\frac{\overline{r}}{2}$$

A[2]

SeriesSolve::NoSolution: No solution in degree 2.

\$Aborted

```
ks[ks___] := ks[ks] = SeriesCoefficient[ $\frac{1}{4} \text{Log}\left[\frac{\hbar/2}{\text{Sinh}[\hbar/2]}\right]$ , {ħ, 0, Length@{ks}}];
```

```
κ = CWS[{x}, ks];
```

```
Cap0 = Es[⟨x → LS[0]⟩, κ];
```

κ@{16}

$$\text{CWS}\left[0, -\frac{\overline{xx}}{96}, 0, \frac{\overline{xxxx}}{11520}, 0, -\frac{\overline{xxxxxx}}{725760}, 0, \frac{\overline{xxxxxxxx}}{38707200}, 0, -\frac{\overline{xxxxxxxxxx}}{1916006400}, 0, \frac{691 \overline{xxxxxxxxxxxx}}{62768369664000}, 0, -\frac{\overline{xxxxxxxxxxxxxxxx}}{4184557977600}, 0, \frac{3617 \overline{xxxxxxxxxxxxxxxxxx}}{682919861944320000}, \dots\right]$$

```
SymmetryFactor[l_List] := Module[{ll = l, sf = 1},
```

```
While[(ll = RotateLeft[ll]) != l, ++sf];
```

```
sf];
```

```
SymmetryFactor[{1, 1, 1, 2, 1, 1, 1, 2}]
```

4

```
γs[ks___] :=
```

```
γs[ks] = If[{ks} === RotateLeft@{ks}, 0, SymmetryFactor[{ks}]] * ks[ks];
```

```
γ = CWS[{x, y}, γs];
```

γ@{6}

$$\text{CWS}\left[0, -\frac{\overline{xy}}{48}, 0, \frac{\overline{xxxxy}}{2880} + \frac{\overline{xyyy}}{2880} + \frac{\overline{xyxy}}{5760} + \frac{\overline{xyyy}}{2880}, 0, -\frac{\overline{xxxxxy}}{120960} - \frac{\overline{xxxxyy}}{120960} - \frac{\overline{xxxxyx}}{120960} - \frac{\overline{xxxyyy}}{120960} - \frac{\overline{xyxxxxy}}{241920} - \frac{\overline{xyxyxy}}{120960} - \frac{\overline{xyyyxy}}{120960} - \frac{\overline{xyyyyy}}{362880} - \frac{\overline{xyxyxy}}{120960} - \frac{\overline{xyyyxy}}{241920} - \frac{\overline{xyyyyy}}{120960}, \dots\right]$$

```

βs[t_]["x"] := t; βs[t_]["y"] := 0;
α[t_] := α[t] = LS[{x, y}, as[t]]; β[t_] := β[t] = LS[{x, y}, βs[t]];
V[t_] := (
  V[t] = Es[⟨x → α[t], y → β[t]⟩, γ];
  SeriesSolve[{α[t], β[t]},
    (ħ⁻¹ R4Eqn[V[t]]) ∧ UnitarityEqn[V[t]] ∧ CapEqn[V[t], Cap₀];
  V[t]
);
V[1/2]@{6}

```

$$\begin{aligned}
 & \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{x\overline{y}}}{24}, 0, \frac{7 \sqrt{x \overline{x\overline{y}}}}{5760} - \frac{7 \sqrt{x \overline{x\overline{y}y}}}{5760} + \frac{\sqrt{\overline{x\overline{y}y}}}{1440}, 0, \right. \right. \right. \\
 & \quad - \frac{31 \sqrt{x \overline{x \overline{x\overline{y}}}}}{967680} + \frac{31 \sqrt{x \overline{x \overline{x\overline{y}y}}}}{483840} - \frac{83 \sqrt{x \overline{x \overline{x\overline{y}y}}}}{967680} - \frac{31 \sqrt{x \overline{x\overline{y}} \overline{x\overline{y}y}}}{725760} - \frac{31 \sqrt{x \overline{x \overline{x\overline{y}} \overline{x\overline{y}}}}}{645120} + \\
 & \quad \left. \left. \left. \frac{13 \sqrt{x \overline{x\overline{y}y} \overline{y}}}{241920} + \frac{101 \sqrt{x \overline{x\overline{y}} \overline{x\overline{y}y}}}{1451520} + \frac{527 \sqrt{x \overline{x\overline{y}y} \overline{x\overline{y}}}}{5806080} - \frac{\sqrt{\overline{x\overline{y}y} \overline{y} \overline{y}}}{60480}, \dots \right] \right), \\
 & \quad \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{x\overline{y}}}{12}, 0, \frac{\sqrt{x \overline{x\overline{y}}}}{5760} - \frac{1}{720} \sqrt{x \overline{x\overline{y}y}} + \frac{1}{720} \sqrt{\overline{x\overline{y}y} \overline{y}}, -\frac{\sqrt{x \overline{x \overline{x\overline{y}}}}}{7680} + \frac{\sqrt{x \overline{x \overline{x\overline{y}y}}}}{3840} - \frac{\sqrt{\overline{x \overline{x\overline{y}} \overline{x\overline{y}}}}}{6912}, \right. \\
 & \quad - \frac{\sqrt{x \overline{x \overline{x\overline{y}}}}}{645120} + \frac{23 \sqrt{x \overline{x \overline{x\overline{y}y}}}}{483840} - \frac{13 \sqrt{x \overline{x \overline{x\overline{y}y}}}}{161280} - \frac{\sqrt{x \overline{x\overline{y}} \overline{x\overline{y}y}}}{22680} - \frac{41 \sqrt{x \overline{x \overline{x\overline{y}} \overline{x\overline{y}}}}}{580608} + \\
 & \quad \left. \left. \left. \frac{\sqrt{x \overline{x\overline{y}y} \overline{y}}}{15120} + \frac{\sqrt{\overline{x\overline{y}} \overline{x\overline{y}y}}}{12096} + \frac{71 \sqrt{x \overline{x\overline{y}y} \overline{x\overline{y}}}}{483840} - \frac{\sqrt{\overline{x\overline{y}y} \overline{y} \overline{y}}}{30240}, \dots \right] \right), \\
 & \quad \text{CWS} \left[0, -\frac{\overline{x\overline{y}}}{48}, 0, \frac{\overline{xxx\overline{y}}}{2880} + \frac{\overline{xx\overline{y}y}}{2880} + \frac{\overline{xyx\overline{y}}}{5760} + \frac{\overline{xy\overline{y}y}}{2880}, 0, \right. \\
 & \quad - \frac{\overline{xxxxx\overline{y}}}{120960} - \frac{\overline{xxxx\overline{y}y}}{120960} - \frac{\overline{xxx\overline{y}x\overline{y}}}{120960} - \frac{\overline{xxx\overline{y}y\overline{y}}}{120960} - \frac{\overline{xx\overline{y}x\overline{y}}}{241920} - \frac{\overline{xx\overline{y}y\overline{y}}}{120960} - \\
 & \quad \left. \frac{\overline{xx\overline{y}y\overline{y}x\overline{y}}}{120960} - \frac{\overline{xx\overline{y}y\overline{y}y\overline{y}}}{120960} - \frac{\overline{x\overline{y}x\overline{y}x\overline{y}}}{362880} - \frac{\overline{x\overline{y}x\overline{y}y\overline{y}}}{120960} - \frac{\overline{x\overline{y}y\overline{y}x\overline{y}}}{241920} - \frac{\overline{x\overline{y}y\overline{y}y\overline{y}}}{120960}, \dots \right]
 \end{aligned}$$

```

τ[V_] = Rs[x, y] ** (V // dσ[{x, y} → {y, x}]) ** θs[x, y, -1/2];
(V[#] ≡ τ[V[#]])@{6} & /@ {1/4, 1/3, 1/2, 1}
{BS[7 True, ...], BS[7 True, ...], BS[7 True, ...], BS[7 True, ...]}

```

Relations

```

(V[0] ** dA[V[1]])@{6}

```

$$\begin{aligned}
 & \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[-\overline{y}, \overline{x\overline{y}}, -\frac{1}{2} \sqrt{x \overline{x\overline{y}}}, \frac{1}{6} \sqrt{x \overline{x \overline{x\overline{y}}}}, -\frac{1}{24} \sqrt{x \overline{x \overline{x \overline{x\overline{y}}}}, \frac{1}{120} \sqrt{x \overline{x \overline{x \overline{x \overline{x\overline{y}}}}}, \dots \right], \right. \right. \\
 & \quad \left. \left. \overline{y} \rightarrow \text{LS} \left[-\overline{x}, 0, 0, 0, 0, 0, \dots \right] \right), \text{CWS} \left[0, 0, 0, 0, 0, 0, \dots \right] \right]
 \end{aligned}$$

$(\mathbf{v}[0] ** \mathbf{ds}[\mathbf{v}[1]]) @ \{6\}$

Es[$\langle \overline{x} \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots], \overline{y} \rightarrow \text{LS}[\overline{x}, 0, 0, 0, 0, 0, \dots] \rangle$,
CWS[0, 0, 0, 0, 0, 0, ...]]

$(\mathbf{v}[1/4] \equiv (\mathbf{v}[1/4] // \mathbf{ds}[\{\mathbf{x}, \mathbf{y}\} \rightarrow \{\mathbf{y}, \mathbf{x}\}] // (-1)^{\text{deg}})) @ \{8\}$

SeriesSolve::ArbitrarilySetting: In degree 8 arbitrarily setting $\{\alpha s[\frac{1}{4}[x, x, x, x, y, x, y, y] \rightarrow 0\}$.

BS[9 True, ...]

$((\mathbf{v}[1/4] // \mathbf{dA}) ** (\mathbf{v}[1/4] // \mathbf{ds}[\{\mathbf{x}, \mathbf{y}\} \rightarrow \{\mathbf{y}, \mathbf{x}\}] // (-1)^{\text{deg}})) @ \{6\}$

Es[$\langle \overline{x} \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots], \overline{y} \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots] \rangle$,
CWS[0, 0, 0, 0, 0, 0, ...]]

$(\mathbf{v}[1/2] ** (\mathbf{v}[1/2] // \mathbf{ds})) @ \{6\}$

Es[$\langle \overline{x} \rightarrow \text{LS}[0, 0, 0, 0, 0, 0, \dots], \overline{y} \rightarrow \text{LS}[\overline{x}, 0, 0, 0, 0, 0, \dots] \rangle$,
CWS[0, 0, 0, 0, 0, 0, ...]]

$((\mathbf{v}[1/4] ** \mathbf{es}[\mathbf{x}, \mathbf{y}, 1/12]) \equiv \mathbf{v}[1/3]) @ \{7\}$

BS[8 True, ...]

Play

$\{\mathbf{v}[0], \mathbf{v}[1/4], \mathbf{v}[1/2], \mathbf{v}[3/4], \mathbf{v}[1]\} // \mathbf{ColumnForm}$

Es[$\langle \overline{x} \rightarrow \text{LS}[-\frac{\overline{y}}{2}, \frac{\overline{x}\overline{y}}{12}, 0, \dots], \overline{y} \rightarrow \text{LS}[0, \frac{\overline{x}\overline{y}}{24}, 0, \dots] \rangle$, CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, ...]]
 Es[$\langle \overline{x} \rightarrow \text{LS}[-\frac{\overline{y}}{4}, -\frac{\overline{x}\overline{y}}{96}, \frac{1}{128} \overline{x\overline{x}\overline{y}} - \frac{1}{256} \overline{x\overline{y}\overline{y}}, \dots], \overline{y} \rightarrow \text{LS}[\frac{\overline{x}}{4}, \frac{\overline{x}\overline{y}}{96}, \frac{1}{256} \overline{x\overline{x}\overline{y}} - \frac{1}{128} \overline{x\overline{y}\overline{y}}, \dots] \rangle$, CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, ...]]
 Es[$\langle \overline{x} \rightarrow \text{LS}[0, -\frac{\overline{x}\overline{y}}{24}, 0, \dots], \overline{y} \rightarrow \text{LS}[\frac{\overline{x}}{2}, -\frac{\overline{x}\overline{y}}{12}, 0, \dots] \rangle$, CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, ...]]
 Es[$\langle \overline{x} \rightarrow \text{LS}[\frac{\overline{y}}{4}, -\frac{\overline{x}\overline{y}}{96}, -\frac{1}{128} \overline{x\overline{x}\overline{y}} + \frac{1}{256} \overline{x\overline{y}\overline{y}}, \dots], \overline{y} \rightarrow \text{LS}[\frac{3\overline{x}}{4}, -\frac{23\overline{x}\overline{y}}{96}, -\frac{5}{256} \overline{x\overline{x}\overline{y}} + \frac{5}{128} \overline{x\overline{y}\overline{y}}, \dots] \rangle$, CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, ...]]
 Es[$\langle \overline{x} \rightarrow \text{LS}[\frac{\overline{y}}{2}, \frac{\overline{x}\overline{y}}{12}, 0, \dots], \overline{y} \rightarrow \text{LS}[\overline{x}, -\frac{11\overline{x}\overline{y}}{24}, -\frac{1}{16} \overline{x\overline{x}\overline{y}} + \frac{1}{8} \overline{x\overline{y}\overline{y}}, \dots] \rangle$, CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, ...]]

$\mathbf{v}[1/2] @ \{4\}$

Es[$\langle \overline{x} \rightarrow \text{LS}[0, -\frac{\overline{x}\overline{y}}{24}, 0, \frac{7 \overline{x\overline{x}\overline{x}\overline{y}}}{5760} - \frac{7 \overline{x\overline{x}\overline{y}\overline{y}}}{5760} + \frac{\overline{x\overline{y}\overline{y}\overline{y}}}{1440}, \dots] \rangle$,
 $\overline{y} \rightarrow \text{LS}[\frac{\overline{x}}{2}, -\frac{\overline{x}\overline{y}}{12}, 0, \frac{\overline{x\overline{x}\overline{x}\overline{y}}}{5760} - \frac{1}{720} \overline{x\overline{x}\overline{y}\overline{y}} + \frac{1}{720} \overline{x\overline{y}\overline{y}\overline{y}}, \dots] \rangle$,
 CWS[0, $-\frac{\overline{x}\overline{y}}{48}$, 0, $\frac{\overline{x\overline{x}\overline{x}\overline{y}}}{2880} + \frac{\overline{x\overline{x}\overline{y}\overline{y}}}{2880} + \frac{\overline{x\overline{y}\overline{x}\overline{y}}}{5760} + \frac{\overline{x\overline{y}\overline{y}\overline{y}}}{2880}, \dots]$]

(dA[v[0]] ** v[1]) @ {5}

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[\overline{y}, \frac{\overline{x\overline{y}}}{2}, \frac{1}{6} \overline{x\overline{x\overline{y}}} - \frac{1}{12} \overline{x\overline{y\overline{y}}}, \frac{1}{24} \overline{x\overline{x\overline{x\overline{y}}}} - \frac{1}{24} \overline{x\overline{x\overline{y\overline{y}}}}, \frac{1}{120} \overline{x\overline{x\overline{x\overline{x\overline{y}}}}} - \right. \right. \\
 \left. \left. \frac{1}{80} \overline{x\overline{x\overline{x\overline{y\overline{y}}}}} + \frac{1}{720} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} + \frac{1}{180} \overline{x\overline{y\overline{x\overline{y\overline{y}}}}} + \frac{1}{240} \overline{x\overline{x\overline{y\overline{x\overline{y}}}}} + \frac{1}{720} \overline{x\overline{y\overline{y\overline{y\overline{y}}}}}, \dots \right], \right. \\
 \left. \overline{y} \rightarrow \text{LS} \left[\overline{x}, -\frac{\overline{x\overline{y}}}{2}, -\frac{1}{12} \overline{x\overline{x\overline{y}}} + \frac{1}{6} \overline{x\overline{y\overline{y}}}, \frac{1}{24} \overline{x\overline{x\overline{y\overline{y}}}} - \frac{1}{24} \overline{x\overline{y\overline{y\overline{y}}}}, \right. \right. \\
 \left. \left. \frac{1}{720} \overline{x\overline{x\overline{x\overline{x\overline{y}}}}} + \frac{1}{720} \overline{x\overline{x\overline{x\overline{y\overline{y}}}}} - \frac{1}{80} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} - \frac{1}{120} \overline{x\overline{y\overline{x\overline{y\overline{y}}}}} + \frac{1}{240} \overline{x\overline{x\overline{y\overline{x\overline{y}}}}} + \frac{1}{120} \overline{x\overline{y\overline{y\overline{y\overline{y}}}}}, \right. \right. \\
 \left. \left. \dots \right] \right), \text{CWS}[0, 0, 0, 0, 0, \dots]]
 \end{aligned}$$

Series[e^{h/2}, {h, 0, 8}]

$$1 + \frac{h}{2} + \frac{h^2}{8} + \frac{h^3}{48} + \frac{h^4}{384} + \frac{h^5}{3840} + \frac{h^6}{46080} + \frac{h^7}{645120} + \frac{h^8}{10321920} + O[h]^9$$

(ds[v[0]] ** v[1]) @ {5}

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{x\overline{y}}}{2}, -\frac{1}{3} \overline{x\overline{x\overline{y}}} + \frac{1}{8} \overline{x\overline{y\overline{y}}}, -\frac{1}{8} \overline{x\overline{x\overline{x\overline{y}}}} + \frac{1}{12} \overline{x\overline{x\overline{y\overline{y}}}} - \frac{1}{48} \overline{x\overline{y\overline{y\overline{y}}}}, \right. \right. \\
 \left. \left. -\frac{1}{30} \overline{x\overline{x\overline{x\overline{x\overline{y}}}}} + \frac{43}{1440} \overline{x\overline{x\overline{x\overline{y\overline{y}}}}} - \frac{79}{5760} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} - \frac{5}{288} \overline{x\overline{x\overline{y\overline{x\overline{y}}}}} + \frac{1}{384} \overline{x\overline{y\overline{y\overline{y\overline{y}}}}}, \dots \right], \right. \\
 \left. \overline{y} \rightarrow \text{LS} \left[\overline{x}, -\frac{\overline{x\overline{y}}}{2}, -\frac{1}{12} \overline{x\overline{x\overline{y}}} + \frac{1}{8} \overline{x\overline{y\overline{y}}}, \frac{1}{48} \overline{x\overline{x\overline{y\overline{y}}}} - \frac{1}{48} \overline{x\overline{y\overline{y\overline{y}}}}, \frac{1}{720} \overline{x\overline{x\overline{x\overline{x\overline{y}}}}} - \frac{1}{720} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} - \right. \right. \\
 \left. \left. \frac{19}{5760} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} + \frac{1}{288} \overline{x\overline{x\overline{y\overline{x\overline{y}}}}} + \frac{1}{384} \overline{x\overline{y\overline{y\overline{y\overline{y}}}}}, \dots \right] \right), \text{CWS}[0, 0, 0, 0, 0, \dots]]
 \end{aligned}$$

(v[0] ** ds[v[1/2]]) @ {6}

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{2}, -\frac{\overline{x\overline{y}}}{8}, -\frac{1}{32} \overline{x\overline{y\overline{y}}}, \frac{1}{384} \overline{x\overline{x\overline{x\overline{y}}}} - \frac{1}{128} \overline{x\overline{x\overline{y\overline{y}}}}, \right. \right. \\
 \left. \left. \frac{\overline{x\overline{x\overline{x\overline{y\overline{y}}}}}{1536} - \frac{1}{512} \overline{x\overline{x\overline{y\overline{y\overline{y}}}}} - \frac{1}{384} \overline{x\overline{y\overline{x\overline{y\overline{y}}}}} - \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{1536} + \frac{\overline{x\overline{y\overline{y\overline{y\overline{y}}}}}{1536}, \right. \right. \\
 \left. \left. -\frac{\overline{x\overline{x\overline{x\overline{x\overline{y}}}}}{15360} + \frac{\overline{x\overline{x\overline{x\overline{y\overline{y}}}}}{3072} - \frac{\overline{x\overline{x\overline{y\overline{y\overline{y}}}}}{2048} - \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{1536} - \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{6144} + \frac{\overline{x\overline{y\overline{y\overline{y\overline{y}}}}}{6144}, \dots \right], \right. \\
 \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, \frac{\overline{x\overline{y}}}{8}, \frac{1}{32} \overline{x\overline{x\overline{y}}}, \frac{1}{128} \overline{x\overline{x\overline{y\overline{y}}}} - \frac{1}{384} \overline{x\overline{y\overline{y\overline{y}}}}, -\frac{\overline{x\overline{x\overline{x\overline{y}}}}}{1536} + \frac{1}{512} \overline{x\overline{x\overline{y\overline{y}}}} - \right. \right. \\
 \left. \left. \frac{\overline{x\overline{x\overline{y\overline{y\overline{y}}}}}{1536} + \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{1536}, -\frac{\overline{x\overline{x\overline{x\overline{y\overline{y}}}}}{6144} + \frac{\overline{x\overline{x\overline{y\overline{y\overline{y}}}}}{2048} + \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{1536} + \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{3072} - \frac{\overline{x\overline{y\overline{y\overline{y\overline{y}}}}}{3072} - \right. \right. \\
 \left. \left. \frac{\overline{x\overline{y\overline{x\overline{y\overline{y}}}}}{2048} - \frac{\overline{x\overline{x\overline{y\overline{x\overline{y}}}}}{6144} + \frac{\overline{x\overline{y\overline{y\overline{y\overline{y}}}}}{15360}, \dots \right] \right), \text{CWS}[0, 0, 0, 0, 0, 0, \dots]]
 \end{aligned}$$

v[1/4]

$$\text{Es} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{4}, -\frac{\overline{xy}}{96}, \frac{1}{128} \overline{xx\overline{y}} - \frac{1}{256} \overline{xy\overline{y}}, \dots \right], \right. \right. \\ \left. \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{4}, \frac{\overline{xy}}{96}, \frac{1}{256} \overline{xx\overline{y}} - \frac{1}{128} \overline{xy\overline{y}}, \dots \right] \right\rangle, \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

v[1/4] // dσ[{x, y} → {y, x}] // (-1)^{deg}

$$\text{Es} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{4}, -\frac{\overline{xy}}{96}, \frac{1}{128} \overline{xx\overline{y}} - \frac{1}{256} \overline{xy\overline{y}}, \dots \right], \right. \right. \\ \left. \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{4}, \frac{\overline{xy}}{96}, \frac{1}{256} \overline{xx\overline{y}} - \frac{1}{128} \overline{xy\overline{y}}, \dots \right] \right\rangle, \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

v[1/2]

$$\text{Es} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{xy}}{24}, 0, \dots \right], \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \dots \right] \right\rangle, \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

v[1/2] // dσ[{x, y} → {y, x}] // (-1)^{deg}

$$\text{Es} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, 0, \dots \right], \overline{y} \rightarrow \text{LS} \left[0, \frac{\overline{xy}}{24}, 0, \dots \right] \right\rangle, \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

((v[0] // dA) ** (v[0] // dσ[{x, y} → {y, x}] // (-1)^{deg})) @ {6}

$$\text{Es} \left[\left\langle \overline{x} \rightarrow \text{LS} \left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{8}, \frac{1}{48} \overline{xx\overline{y}} - \frac{1}{96} \overline{xy\overline{y}}, \frac{1}{384} \overline{xxx\overline{y}} - \frac{1}{384} \overline{xx\overline{yy}}, \right. \right. \\ \frac{\overline{xxx\overline{xy}}}{3840} - \frac{\overline{xxx\overline{xyy}}}{2560} + \frac{\overline{xx\overline{xyyy}}}{23040} + \frac{\overline{xy\overline{xyy}}}{5760} + \frac{\overline{xx\overline{xyxy}}}{7680} + \frac{\overline{xy\overline{yyyy}}}{23040}, \\ \frac{\overline{xxx\overline{xyy}}}{46080} - \frac{\overline{xxx\overline{xyyy}}}{23040} + \frac{\overline{xx\overline{xyyy}}}{92160} + \frac{\overline{xx\overline{xyxy}}}{23040} + \frac{\overline{xx\overline{xyxy}}}{30720} + \frac{\overline{xx\overline{xyyy}}}{92160}, \dots \left. \right], \\ \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{8}, -\frac{1}{96} \overline{xx\overline{y}} + \frac{1}{48} \overline{xy\overline{y}}, \frac{1}{384} \overline{xxx\overline{y}} - \frac{1}{384} \overline{xx\overline{yy}}, \right. \\ \frac{\overline{xxx\overline{xy}}}{23040} + \frac{\overline{xxx\overline{xyy}}}{23040} - \frac{\overline{xx\overline{xyyy}}}{2560} - \frac{\overline{xy\overline{xyy}}}{3840} + \frac{\overline{xx\overline{xyxy}}}{7680} + \frac{\overline{xy\overline{yyyy}}}{3840}, \\ -\frac{\overline{xxx\overline{xyy}}}{92160} - \frac{\overline{xxx\overline{xyyy}}}{92160} - \frac{\overline{xx\overline{xyxy}}}{23040} + \frac{\overline{xx\overline{xyxy}}}{46080} + \frac{\overline{xx\overline{xyyy}}}{23040} + \\ \left. \frac{\overline{xy\overline{xyyy}}}{18432} - \frac{\overline{xx\overline{xyxy}}}{18432} - \frac{\overline{xy\overline{yyyy}}}{46080}, \dots \right\rangle, \text{CWS} \left[0, 0, 0, 0, 0, 0, \dots \right] \right]$$

(v[1/4] ** (v[1/4] // ds)) @ {6}

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{2}, -\frac{\overline{xy}}{8}, -\frac{1}{32} \overline{xyy}, \frac{1}{384} \overline{xxxy} - \frac{1}{128} \overline{xyxy}, \right. \right. \right. \\
 \frac{\overline{xxxyy}}{1536} - \frac{1}{512} \overline{xyxyy} - \frac{1}{384} \overline{xyxyy} - \frac{\overline{xyxyxy}}{1536} + \frac{\overline{xyyyyy}}{1536}, \\
 \left. \left. -\frac{\overline{xxxxy}}{15360} + \frac{\overline{xxxxy}}{3072} - \frac{\overline{xyxyy}}{2048} - \frac{\overline{xyxyxy}}{1536} - \frac{\overline{xyxyxy}}{6144} + \frac{\overline{xyxyyy}}{6144}, \dots \right] \right), \\
 \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, \frac{\overline{xy}}{8}, \frac{1}{32} \overline{xyxy}, \frac{1}{128} \overline{xyxy} - \frac{1}{384} \overline{xyxyy}, -\frac{\overline{xxxy}}{1536} + \frac{1}{512} \overline{xxxy} - \right. \\
 \frac{\overline{xyxyy}}{1536} + \frac{\overline{xyxyxy}}{1536}, -\frac{\overline{xxxxy}}{6144} + \frac{\overline{xyxyy}}{2048} + \frac{\overline{xyxyxy}}{1536} + \frac{\overline{xyxyxy}}{3072} - \frac{\overline{xyxyyy}}{3072} \\
 \left. \left. - \frac{\overline{xyxyxy}}{2048} - \frac{\overline{xyxyxy}}{6144} + \frac{\overline{xyxyyy}}{15360}, \dots \right] \right), \text{CWS}[0, 0, 0, 0, 0, 0, \dots]
 \end{aligned}$$

Os[x, y, 1/2]

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{8}, \frac{1}{48} \overline{xxxy} - \frac{1}{96} \overline{xyxy}, \dots \right], \right. \right. \\
 \left. \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{8}, -\frac{1}{96} \overline{xxxy} + \frac{1}{48} \overline{xyxy}, \dots \right] \right), \text{CWS}[0, 0, 0, \dots]
 \end{aligned}$$

v[s]

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[\frac{1}{2} (-1 + 2s) \overline{y}, \frac{1}{12} (1 - 6s + 6s^2) \overline{xy}, \right. \right. \right. \\
 \left. \left. \frac{1}{12} (s - 3s^2 + 2s^3) \overline{xxxy} + \frac{1}{24} (-s + 3s^2 - 2s^3) \overline{xyxy}, \dots \right] \right), \\
 \left. \overline{y} \rightarrow \text{LS} \left[s \overline{x}, \frac{1}{24} (1 - 12s^2) \overline{xy}, \frac{1}{48} (s - 4s^3) \overline{xxxy} + \frac{1}{24} (-s + 4s^3) \overline{xyxy}, \dots \right] \right), \\
 \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right]
 \end{aligned}$$

v[1/2]

$$\text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{xy}}{24}, 0, \dots \right], \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \dots \right] \right), \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

v[1/4] ** Os[x, y, 1/4]

$$\text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[0, -\frac{\overline{xy}}{24}, 0, \dots \right], \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, 0, \dots \right] \right), \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]$$

v[1/3]

$$\begin{aligned}
 \text{Es} \left[\left(\overline{x} \rightarrow \text{LS} \left[-\frac{\overline{y}}{6}, -\frac{\overline{xy}}{36}, \frac{1}{162} \overline{xxxy} - \frac{1}{324} \overline{xyxy}, \dots \right], \right. \right. \\
 \left. \left. \overline{y} \rightarrow \text{LS} \left[\frac{\overline{x}}{3}, -\frac{\overline{xy}}{72}, \frac{5 \overline{xxxy}}{1296} - \frac{5}{648} \overline{xyxy}, \dots \right] \right), \text{CWS} \left[0, -\frac{\overline{xy}}{48}, 0, \dots \right] \right]
 \end{aligned}$$

