A Proof of the Cayley-Hamilton Theorem.

Dror Bar-Natan, December 30, 2015.

Theorem. Let R be a commutative ring, let $A \in M_{n\times n}(R)$ be a matrix and let $\chi_A(t) \coloneqq \det(tI - A)$ be the characteristic polynomial of A. Then $\chi_A(A) = 0$.

Proof. For any matrix M over any commutative ring there is "the adjoint matrix $\operatorname{adj}(M)$ of M", defined using the minors of M, which satisfies $\det(M)I = (\operatorname{adj} M)M$. Use this with $M = tI - A$, over the ring $R[t]$, and find that in the ring $M_{n\times n}(R[t])$ we have

$$\chi_A(t)I = \det(tI - A)I = (\operatorname{adj}(tI - A))(tI - A).$$

Now note that the ring $M_{n\times n}(R[t])$ is isomorphic to the ring $M_{n\times n}(R)[t]$, and on the latter there is a linear "evaluation at $t = A$" map $ev_A \colon M_{n\times n}(R)[t] \to M_{n\times n}(R)$, defined by "putting A to the right of all the coefficients"; namely, by $\sum B_k t^k \mapsto \sum B_k A^k$. This evaluation map ev_A is *not* multiplicative, but nevertheless it annihilates anything that has a right factor of $(tI - A)$. Hence under ev_A the above equality becomes

$$\chi_A(A)I = 0.$$

□