Tingley: Crystal combinatorics from PBW bases

Monday, November 23, 2015 4:15 PM

1. PBW bases for $U_q^-(\mathfrak{g})$
2. Equality mod q $\rightsquigarrow$ Crystals.
3. bar involution and canonical basis.

PBW bases; $\mathfrak{g}$ of type ADE.

$U_q(\mathfrak{g}) = \langle E_i, F_i, K_i \rangle / \text{relations}$.

$U_q^-(\mathfrak{g}) = \langle F_i \rangle$

Define T_i by

$$T_i(F_j) = \begin{cases} F_j & \text{if } a_{ij} = 0 \\ F_j F_i - q F_i F_j & a_{ij} = -1 \\ -K_j E_i & i = j \end{cases}$$

Algebra automorphisms satisfying the braid relations

$$T_i T_j T_i = T_j T_i T_j \quad \text{if } i - j$$
$$T_i T_j = T_j T_i \quad \text{if } i \not- j$$

Fix $w_0 = s_{i_1} \cdots s_{i_N}$ reduced expression for w_0

then

$$F_{\beta_1} = F_{i_1} \qquad \beta_1 = \alpha_{i_1}$$
$$F_{\beta_2} = T_{i_1} F_{i_2} \qquad \beta_2 = s_{i_1} \alpha_{i_2}.$$

$F_{\beta_3} = T_{i_1} T_{i_2} F_{i_3}$ wts are neg roots

Thm (Lusztig) $\vec{\imath} = (i_1 \dots i_N)$

i.) $F_\beta \in U_q^-(g)$

ii) $B^{\vec{\imath}} = \{F_{\beta_1}^{(a_1)} \dots F_{\beta_N}^{(a_N)} : a_1 \dots a_N \geq 0\}$ is a basis of $U_q^-(g)$

iii) $\mathcal{L} = \mathrm{span}_{\mathbb{Z}[q]} B^{\vec{\imath}}$ does not depend on $\vec{\imath}$.

iv) $B = B^{\vec{\imath}} + q\mathcal{L} \subset \mathcal{L}/q\mathcal{L}$ does not depend on $\vec{\imath}$.

"Proof"

* Watch what happens to $B^{\vec{\imath}}$ when $\vec{\imath}$ changes by a single braid move. 0:23

Aside $12312\,1 = 123212 = 132312 = 312132 =$

$= 321232 = 321323$