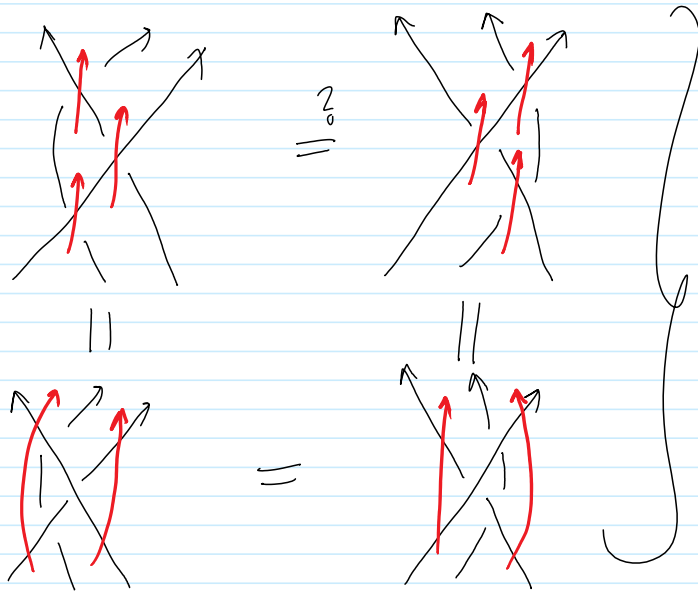


Crossing the crossings - Post Mortem

Wednesday, November 4, 2015 9:19 AM



I should have included the braid-like R3 argument,

Dror Bar-Natan: Talks: MoscowByWeb-1511:

Video and more at <https://drorbn.net/mbw>

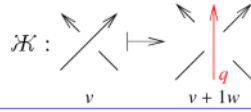
Crossing the Crossings



Abstract. The subject will be very close to Manturov's representation of vB_n into $\text{Aut}(FG_{n+1})$ — I'll describe how I think about it in terms of a very simple minded map \mathcal{K} from n -component v -tangles to $(n+1)$ -component w -tangles. It is possible that you all know this already. Possibly my talk will be very short — it will be as long as it is necessary to describe \mathcal{K} and say a few more words, and if this is little, so be it.

All you need is \mathcal{K} ...

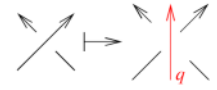
- What is its domain?
- What is its target?
- Why should one care?



Back to \mathcal{K} . The “crossing the crossings” map $\mathcal{K}: vT_n \rightarrow wT_{n+1}$ is defined by the picture below. Equally well, it is $\mathcal{K}: vB_n \rightarrow wB_{n+1}$. Better, it is $\mathcal{K}: vT_n \rightarrow (nv+1w)T$ or $\mathcal{K}: vB_n \rightarrow (nv+1w)B$.

Claims.

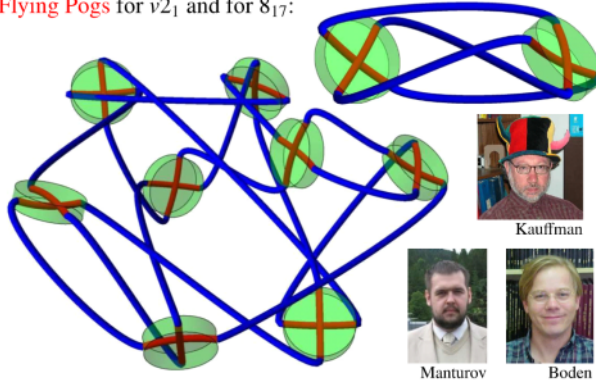
1. \mathcal{K} is well defined.
2. On u -links, \mathcal{K} “factors”.
3. \mathcal{K} does not respect OC .
4. \mathcal{K} recovers Manturov's VG and μ : $VG(K) = \pi_1(\mathcal{K}(K))$, $\mu = \mathcal{K} \circ \phi = \phi // \mathcal{K}$.



Virtual Knots. Virtual knots are the algebraic structure underlying the Reidemeister presentation of ordinary knots, without the topology. Locally they are knot diagrams modulo the Reidemeister relations; globally, who cares? So,

$$vT = CA \langle \times, \times, \times; R1, R2, R3 \rangle \quad CA = \text{“Circuit Algebra”}$$

Flying Pogs for $v2_1$ and for 8_{17} :



No! Note that also (with PA = “Planar Algebra”)

$$vT = PA \langle \times, \times, \times; R1, R2, R3, VR1, VR2, VR3, M \rangle,$$

but I have a prejudice, or a deeply held belief, that **this is morally wrong!**

My moment of reckoning. Manturov's $VG(K)$: [Ma, BGHNW]

$$\begin{array}{ccc} z \times w \rightarrow z = xyx^{-1} & w \times z \rightarrow z = x^{-1}yx & z \times w \rightarrow z = q^{-1}yq \\ x \times y \rightarrow w = x & y \times x \rightarrow w = x & x \times y \rightarrow w = qxq^{-1} \end{array}$$

Manturov's $\mu: vB_n \rightarrow \text{Aut}(F(x_1, \dots, x_n, q))$: [Ma, BGHNW]

$$\sigma_i = \times_i \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \end{cases} \quad \tau_i = \times_i \mapsto \begin{cases} x_i \mapsto q x_{i+1} q^{-1} \\ x_{i+1} \mapsto q^{-1} x_i q \end{cases}$$

Easy resolution. Setting $y_i := q^i x_i q^{-i}$, we find that μ is equivalent to

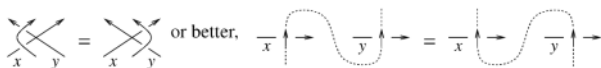
$$\times_i \mapsto \begin{cases} y_i \mapsto y_i q^{-1} y_{i+1} q y_i^{-1} \\ y_{i+1} \mapsto q y_i q^{-1} \end{cases} \quad \times_i \mapsto \begin{cases} y_i \mapsto y_{i+1} \\ y_{i+1} \mapsto y_i \end{cases}$$

and to me, virtual braids are anyways always pure. So really,

$$\sigma_{ij} \mapsto \begin{cases} y_i \mapsto q y_i q^{-1} \\ y_j \mapsto y_i^{-1} q^{-1} y_j q y_i \end{cases}$$

But why **do these** exist? Especially, wherefore $vB_n \rightarrow wB_{n+1}$?

w-Tangles. $wT := vT/OC$ where “Overcrossings Commute” is:

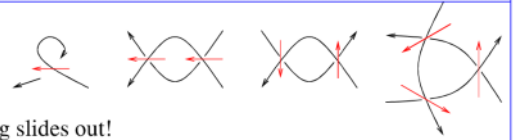


π_1 is defined on wT ; Artin's representation ϕ is defined on wB_n .

Even better, \mathcal{K} pulls back *any* invariant of 2-component w -knots to an invariant of virtual knots. In particular, there is a wheel-valued “non-commutative” invariant ω as in [BN] and [DBN]: Talks: Hamilton-1412 (next page).

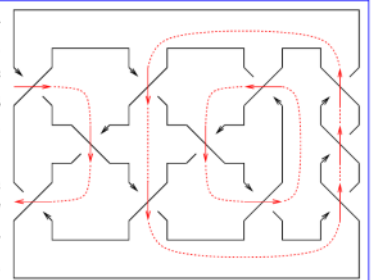
Likely, the various “2-variable Alexander polynomials” for virtual knots arise in this way.

Proof of 1.



Everything slides out!

Proof of 2. The net “red flow” into every **face** is 0, so the red arrows can be paired. They form cycles that can hover off the picture.



region

No proof of 3. Well, there simply is no proof that OC is respected, and it's easy to come up with counter-examples.

Proof of 4. A simple verification, except my conventions are off...

FIX

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, Acta Mathematica Vietnamica **40-2** (2015) 271–329, arXiv:1308.1721.
 [BGHNW] H. U. Boden, A. I. Gaudreau, E. Harper, A. J. Nicas, and L. White, *Virtual Knot Groups and Almost Classical Knots*, arXiv:1506.01726.
 [Ma] V. O. Manturov, *On Invariants of Virtual Links*, Acta Applicandae Mathematica **72-3** (2002) 295–309.

Prejudices should always be re-evaluated!

