Pensieve header: Solving the zipper equations; continues pensieve://2015-09/.

Done:

- The u-involution ui on Γ-calculus: implement, verify invariance of θ, verify homomorphicity.

To do:

- The zipper-twist equation.
- The associator equations.
- The noose equation.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2015-10"];
<< "../Projects/MetaCalculi/MetaCalculi.m"
```

```
MetaCalculi` loading...
```

```
T_i_ := e^(b_i);
bConjugate[expr_] := expr /. b_i_ :→ -b_i;
bSimplify[expr_] :=
  Assuming[b1 > 0 ∧ b2 > 0 ∧ b3 > 0 ∧ bi > 0 ∧ bj > 0, Simplify[PowerExpand[expr]]];
ΓSimp = bSimplify;
```

The u-involution ui on Γ-calculus.

Implement, verify invariance of θ, verify homomorphicity.

```
ui[Γ[ω_, σ_, λ_]] := Module[{S = dL[Γ[ω, σ, λ]], A},
   A = Outer[(∂_{t_#1 h_#2} λ) &, S, S];
   Γ[
     bConjugate[Det[A] * ω / Product[∂_{h_i} σ, {i, S}]],
     σ,
     (h_# / b_# & /@ S).Inverse[bConjugate[A]].(t_# b_# & /@ S)
    ] // ΓSimp
  ];
```

```
{Xp[1, 2] // Γ, Xp[1, 2] // Γ // ui}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-e^{b_1} \\ s_2 & 0 & e^{b_1} \\ \Gamma & 1 & e^{b_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 0 \\ s_2 & -\frac{(-1+e^{b_1})\, b_2}{b_1} & e^{b_1} \\ \Gamma & 1 & e^{b_1} \end{pmatrix}\right\}$$

```
{t1 = Θ[1, 2] // Γ, t2 = Θ[1, 2] // Γ // ui,
 t1 == t2 // bSimplify, (Θ[1, 2] // Γ)@A // Eigenvalues}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{b_1+e^{\frac{1}{2}(b_1+b_2)}\,b_2}{b_1+b_2} & -\frac{\left(-1+e^{\frac{1}{2}(b_1+b_2)}\right)b_1}{b_1+b_2} \\ s_2 & -\frac{\left(-1+e^{\frac{1}{2}(b_1+b_2)}\right)b_2}{b_1+b_2} & \frac{e^{\frac{1}{2}(b_1+b_2)}\,b_1+b_2}{b_1+b_2} \\ \Gamma & \sqrt{e^{b_2}} & \sqrt{e^{b_1}} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{b_1+e^{\frac{1}{2}(b_1+b_2)}\,b_2}{b_1+b_2} & \frac{\left(1-e^{\frac{1}{2}(b_1+b_2)}\right)b_1}{b_1+b_2} \\ s_2 & \frac{\left(1-e^{\frac{1}{2}(b_1+b_2)}\right)b_2}{b_1+b_2} & \frac{e^{\frac{1}{2}(b_1+b_2)}\,b_1+b_2}{b_1+b_2} \\ \Gamma & e^{\frac{b_2}{2}} & e^{\frac{b_1}{2}} \end{pmatrix}, \text{True}, \left\{1, e^{\frac{b_1}{2}+\frac{b_2}{2}}\right\}\right\}$$

$$\left\{n = 3;\ \gamma 0 = \Gamma\left[\omega[b_1, b_2, b_3],\ \sum_{i=1}^{n} h_i \prod_{j=1}^{n} T_j^{\sigma_{10\,i+j}},\ \sum_{i=1}^{n}\sum_{j=1}^{n} t_i\, h_j\, \alpha_{10\,i+j}[b_1, b_2, b_3]\right]\ //\ \text{bSimplify}\right\}$$

$$\left\{\begin{pmatrix} \omega[b_1, b_2, b_3] & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}[b_1, b_2, b_3] & \alpha_{12}[b_1, b_2, b_3] & \alpha_{13}[b_1, b_2, b_3] \\ s_2 & \alpha_{21}[b_1, b_2, b_3] & \alpha_{22}[b_1, b_2, b_3] & \alpha_{23}[b_1, b_2, b_3] \\ s_3 & \alpha_{31}[b_1, b_2, b_3] & \alpha_{32}[b_1, b_2, b_3] & \alpha_{33}[b_1, b_2, b_3] \\ \Gamma & e^{b_1\sigma_{11}+b_2\sigma_{12}+b_3\sigma_{13}} & e^{b_1\sigma_{21}+b_2\sigma_{22}+b_3\sigma_{23}} & e^{b_1\sigma_{31}+b_2\sigma_{32}+b_3\sigma_{33}} \end{pmatrix}\right\}$$

```
{t1 = γ0 // dm[1, 2, 1] // ui,
  t2 = γ0 // ui // dm[1, 2, 1], t1 == t2 // Simplify} // ColumnForm
```

$-e^{b_1(\sigma_{11}+\sigma_{12}+\sigma_{21}+\sigma_{22}+\sigma_{31}+\sigma_{32})+b_3(\sigma_{13}+\sigma_{23}+\sigma_{33})}\,\omega[-b_1, -b_1, -b_3]\,(\alpha_{23}[-b_1, -b_1, -b_3]\,(-(-1+\alpha_{12}[-b_1, -b_1$

$e^{b_1(\sigma_{11}+\sigma_{12}+\sigma_{21}+\sigma_{22}+\sigma_{31}+\sigma_{32})+b_3(\sigma_{13}+\sigma_{23}+\sigma_{33})}\,\omega[-b_1, -b_1, -b_3]\,(\alpha_{23}[-b_1, -b_1, -b_3]\,((-1+\alpha_{12}[-b_1, -b_1, -$

True

```
{t1 = γ0 // dS[1] // ui, t2 = γ0 // ui // dS[1], t1 == t2 // bSimplify} // ColumnForm
```

$-e^{-b_1(\sigma_{21}+\sigma_{31})+b_2(\sigma_{22}+\sigma_{32})+b_3(\sigma_{23}+\sigma_{33})}\,\omega[b_1, -b_2, -b_3]\,(\alpha_{23}[b_1, -b_2, -b_3]\,\alpha_{32}[b_1, -b_2, -b_3]-\alpha_{22}[b_1,$

s_1

s_2

s_3

Γ

$-e^{-b_1(\sigma_{21}+\sigma_{31})+b_2(\sigma_{22}+\sigma_{32})+b_3(\sigma_{23}+\sigma_{33})}\,\omega[b_1, -b_2, -b_3]\,(\alpha_{23}[b_1, -b_2, -b_3]\,\alpha_{32}[b_1, -b_2, -b_3]-\alpha_{22}[b_1,$

s_1

s_2

s_3

Γ

True

The zipper-twist equation.

```
ϒ0 = Γ[Vi] // bSimplify
```

$$\begin{pmatrix} \frac{((-1+e^{b_1+b_2})\, b_1 b_2)^{1/4}}{((-1+e^{b_1})\,(-1+e^{b_2})\,(b_1+b_2))^{1/4}} & s_1 & s_2 \\ s_1 & \frac{e^{b_2}-e^{b_1+b_2}-e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1\,(b_1+b_2)}}}{1-e^{b_1+b_2}} & \frac{e^{b_2}-e^{b_1+b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}{b_2\,(b_1+b_2)}}}{1-e^{b_1+b_2}} \\ s_2 & \frac{1-e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1\,(b_1+b_2)}}}{1-e^{b_1+b_2}} & -\frac{-1+e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}{b_2\,(b_1+b_2)}}}{1-e^{b_1+b_2}} \\ \Gamma & 1 & e^{-\frac{b_1}{2}} \end{pmatrix}$$

```
Limit[ϒ0[A] /. b_i_ :> ħ b_i, ħ → 0]
```

$$\left\{\left\{\frac{b_1+\sqrt{b_2^2}}{b_1+b_2},\ \frac{b_1-\sqrt{b_1^2}}{b_1+b_2}\right\},\ \left\{\frac{b_2-\sqrt{b_2^2}}{b_1+b_2},\ \frac{\sqrt{b_1^2}+b_2}{b_1+b_2}\right\}\right\}$$

```
θ0 = bSimplify[ϒ0 ** (Xp[1, 2] // Γ) ** (ϒ0 // dA[1, 2] // dσ[1 → 2, 2 → 1])]
```

$$\begin{pmatrix} 1 & s_1 \\ s_1 & \frac{\left(b_1+e^{\frac{1}{2}(b_1+b_2)} b_2\right)\left(e^{b_1+\frac{b_2}{2}}\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})}\, b_2^{3/2}+e^{b_1+\frac{b_2}{2}} b_1\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}+(-1+e^{b_1})(-1+e^{b_1+b_2})\cdots\right.}{\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_1+b_2})}{-1+e^{b_2}}}\,(b_1+b_2)^{3/2}\left(\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}-e^{b_1+\frac{b_2}{2}}\sqrt{b_2(b_1+b_2)}+e^{b_1+\frac{3b_2}{2}}\sqrt{b_2(b_1+b_2)}\right)} \\ s_2 & -\frac{\left(-1+e^{\frac{1}{2}(b_1+b_2)}\right) b_2\left(e^{b_1+\frac{b_2}{2}}\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})}\, b_2^{3/2}+e^{b_1+\frac{b_2}{2}} b_1\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}+(-1+e^{b_1})(-1+e^{b_1+b_2})\cdots\right.}{\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_1+b_2})}{-1+e^{b_2}}}\,(b_1+b_2)^{3/2}\left(\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}-e^{b_1+\frac{b_2}{2}}\sqrt{b_2(b_1+b_2)}+e^{b_1+\frac{3b_2}{2}}\sqrt{b_2(b_1+b_2)}\right)} \\ \Gamma & e^{\frac{b_2}{2}} \end{pmatrix}$$

```
Series[List @@ (θ0 == ui[θ0]) /. a_ == b_ :> a - b /. b_i_ :> ħ b_i, {ħ, 0, 5}] // Normal //
 bSimplify
```

{0, 0, 0, 0, 0}

```
{n = 2; ϒ = Γ[ω[b1, b2], Σ_{i=1}^{n} h_i Π_{j=1}^{n} T_j^{σ_{10 i+j}}, Σ_{i=1}^{n} Σ_{j=1}^{n} t_i h_j α_{10 i+j}[b1, b2]] // bSimplify}
```

$$\left\{\begin{pmatrix} \omega[b_1, b_2] & s_1 & s_2 \\ s_1 & \alpha_{11}[b_1, b_2] & \alpha_{12}[b_1, b_2] \\ s_2 & \alpha_{21}[b_1, b_2] & \alpha_{22}[b_1, b_2] \\ \Gamma & e^{b_1 \sigma_{11}+b_2 \sigma_{12}} & e^{b_1 \sigma_{21}+b_2 \sigma_{22}} \end{pmatrix}\right\}$$

```
θ = ϒ ** (Xp[1, 2] // Γ) ** (ϒ // dS[1, 2] // dσ[1 → 2, 2 → 1])
```

$$\begin{pmatrix} -e^{b_2(\sigma_{11}+\sigma_{21})+b_1(\sigma_{12}+\sigma_{22})}\,\omega[b_1, b_2]\,\omega[-b_2, -b_1]\,(\alpha_{12}[-b_2, -b_1]\,\alpha_{21}[-b_2, -b_1]-\alpha_{11}[-b_2, -b_1]\,\alpha_{22}[-\cdots \\ s_1 \\ s_2 \\ \Gamma \end{pmatrix}$$

bSimplify[θ == ui[θ]]

$$e^{b_1(\sigma_{11}+\sigma_{21})+b_2(\sigma_{12}+\sigma_{22})}\,\omega[-b_1,-b_2]\,\omega[b_2,b_1]\,(\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]-\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-b_1,-b_2]) == e^{b_2(\sigma_{11}+\sigma_{21})+b_1(\sigma_{12}+\sigma_{22})}\,\omega[b_1,b_2]\,\omega[-b_2,-b_1]\,(\alpha_{12}[-b_2,-b_1]\,\alpha_{21}[-b_2,-b_1]-\alpha_{11}[-b_2,-b_1]\,\alpha_{22}[-b_2,-b_1])\ \&\&$$

$$\left(\alpha_{11}[b_1,b_2]\,\alpha_{11}[-b_2,-b_1]-\alpha_{21}[b_1,b_2]\left(\left(-1+e^{b_1}\right)\alpha_{11}[-b_2,-b_1]+e^{b_1}\alpha_{21}[-b_2,-b_1]\right)\right)\Big/(-\alpha_{12}[-b_2,-b_1]\,\alpha_{21}[-b_2,-b_1]+\alpha_{11}[-b_2,-b_1]\,\alpha_{22}[-b_2,-b_1]) == \left(e^{b_1}\alpha_{12}[-b_1,-b_2]\,\alpha_{12}[b_2,b_1]+\alpha_{22}[-b_1,-b_2]\left(\left(-1+e^{b_1}\right)\alpha_{12}[b_2,b_1]-\alpha_{22}[b_2,b_1]\right)\right)\Big/(\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]-\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-b_1,-b_2])\ \&\&$$

$$\left(-e^{b_1}\alpha_{21}[-b_2,-b_1]\,\alpha_{22}[b_1,b_2]+\alpha_{11}[-b_2,-b_1]\left(\alpha_{12}[b_1,b_2]-\left(-1+e^{b_1}\right)\alpha_{22}[b_1,b_2]\right)\right)\Big/(-\alpha_{12}[-b_2,-b_1]\,\alpha_{21}[-b_2,-b_1]+\alpha_{11}[-b_2,-b_1]\,\alpha_{22}[-b_2,-b_1]) == -\left(\left(b_1\left(e^{b_1}\alpha_{11}[-b_1,-b_2]\,\alpha_{12}[b_2,b_1]+\alpha_{21}[-b_1,-b_2]\left(\left(-1+e^{b_1}\right)\alpha_{12}[b_2,b_1]-\alpha_{22}[b_2,b_1]\right)\right)\right)\Big/(b_2\,(\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]-\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-b_1,-b_2]))\right)\ \&\&$$

$$\left(\alpha_{11}[b_1,b_2]\,\alpha_{12}[-b_2,-b_1]-\alpha_{21}[b_1,b_2]\left(\left(-1+e^{b_1}\right)\alpha_{12}[-b_2,-b_1]+e^{b_1}\alpha_{22}[-b_2,-b_1]\right)\right)\Big/(\alpha_{12}[-b_2,-b_1]\,\alpha_{21}[-b_2,-b_1]-\alpha_{11}[-b_2,-b_1]\,\alpha_{22}[-b_2,-b_1]) == \left(b_2\left(-\alpha_{21}[b_2,b_1]\,\alpha_{22}[-b_1,-b_2]+\alpha_{11}[b_2,b_1]\left(e^{b_1}\alpha_{12}[-b_1,-b_2]+\left(-1+e^{b_1}\right)\alpha_{22}[-b_1,-b_2]\right)\right)\right)\Big/(b_1\,(\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]-\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-b_1,-b_2]))\ \&\&$$

$$\left(\alpha_{12}[b_1,b_2]\,\alpha_{12}[-b_2,-b_1]-\alpha_{22}[b_1,b_2]\left(\left(-1+e^{b_1}\right)\alpha_{12}[-b_2,-b_1]+e^{b_1}\alpha_{22}[-b_2,-b_1]\right)\right)\Big/(\alpha_{12}[-b_2,-b_1]\,\alpha_{21}[-b_2,-b_1]-\alpha_{11}[-b_2,-b_1]\,\alpha_{22}[-b_2,-b_1]) == \left(e^{b_1}\alpha_{11}[-b_1,-b_2]\,\alpha_{11}[b_2,b_1]+\alpha_{21}[-b_1,-b_2]\left(\left(-1+e^{b_1}\right)\alpha_{11}[b_2,b_1]-\alpha_{21}[b_2,b_1]\right)\right)\Big/(-\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]+\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-b_1,-b_2])$$