Pensieve header: Solving the zipper equations; continued pensieve://2015-10/.

Done:

- The u-involution ui on Γ-calculus: implement, verify invariance of θ, verify homomorphicity.

To do:

- The twist equation.
- The associator equations.
- The noose equation.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2015-09"];
<< "../Projects/MetaCalculi/MetaCalculi.m"
```

```
MetaCalculi` loading...
```

```
T_i_ := e^(b_i);
bConjugate[expr_] := expr /. b_i_ :→ -b_i;
bSimplify[expr_] :=
  Assuming[b1 > 0 ⋀ b2 > 0 ⋀ b3 > 0 ⋀ bi > 0 ⋀ bj > 0, Simplify[PowerExpand[expr]]];
ΓSimp = bSimplify;
```

The u-involution ui on Γ-calculus: implement, verify invariance of θ, verify homomorphicity.

```
ui[Γ[ω_, σ_, λ_]] := Module[{S = dL[Γ[ω, σ, λ]], A},
   A = Outer[(∂_{t_#1 h_#2} λ) &, S, S];
   Γ[
     bConjugate[Det[A] * ω / Product[∂_{h_i} σ, {i, S}]],
     σ,
     (h_# / b_# & /@ S).Inverse[bConjugate[A]].(t_# b_# & /@ S)
    ] // ΓSimp
  ];
```

```
{Xp[1, 2] // Γ, Xp[1, 2] // Γ // ui}
```

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1-e^{b_1} \\ s_2 & 0 & e^{b_1} \\ \Gamma & 1 & e^{b_1} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 0 \\ s_2 & -\frac{(-1+e^{b_1})\,b_2}{b_1} & e^{b_1} \\ \Gamma & 1 & e^{b_1} \end{pmatrix}\right\}$$

**{t1 = Θ[1, 2] // Γ, t2 = Θ[1, 2] // Γ // ui,
t1 == t2 // bSimplify, (Θ[1, 2] // Γ)@A // Eigenvalues}**

$$\left\{\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{b_1+e^{\frac{1}{2}(b_1+b_2)}b_2}{b_1+b_2} & -\frac{\left(-1+e^{\frac{1}{2}(b_1+b_2)}\right)b_1}{b_1+b_2} \\ s_2 & -\frac{\left(-1+e^{\frac{1}{2}(b_1+b_2)}\right)b_2}{b_1+b_2} & \frac{e^{\frac{1}{2}(b_1+b_2)}b_1+b_2}{b_1+b_2} \\ \Gamma & \sqrt{e^{b_2}} & \sqrt{e^{b_1}} \end{pmatrix}, \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & \frac{b_1+e^{\frac{1}{2}(b_1+b_2)}b_2}{b_1+b_2} & \frac{\left(1-e^{\frac{1}{2}(b_1+b_2)}\right)b_1}{b_1+b_2} \\ s_2 & \frac{\left(1-e^{\frac{1}{2}(b_1+b_2)}\right)b_2}{b_1+b_2} & \frac{e^{\frac{1}{2}(b_1+b_2)}b_1+b_2}{b_1+b_2} \\ \Gamma & e^{\frac{b_2}{2}} & e^{\frac{b_1}{2}} \end{pmatrix}, \text{True}, \left\{1, e^{\frac{b_1}{2}+\frac{b_2}{2}}\right\}\right\}$$

{n = 3; γ0 = Γ[ω[b₁, b₂, b₃], $\sum_{i=1}^{n} h_i \prod_{j=1}^{n} T_j^{\sigma_{10\,i+j}}$, $\sum_{i=1}^{n}\sum_{j=1}^{n} t_i h_j \alpha_{10\,i+j}$[b₁, b₂, b₃]] // bSimplify}

$$\left\{\begin{pmatrix} \omega[b_1,b_2,b_3] & s_1 & s_2 & s_3 \\ s_1 & \alpha_{11}[b_1,b_2,b_3] & \alpha_{12}[b_1,b_2,b_3] & \alpha_{13}[b_1,b_2,b_3] \\ s_2 & \alpha_{21}[b_1,b_2,b_3] & \alpha_{22}[b_1,b_2,b_3] & \alpha_{23}[b_1,b_2,b_3] \\ s_3 & \alpha_{31}[b_1,b_2,b_3] & \alpha_{32}[b_1,b_2,b_3] & \alpha_{33}[b_1,b_2,b_3] \\ \Gamma & e^{b_1\sigma_{11}+b_2\sigma_{12}+b_3\sigma_{13}} & e^{b_1\sigma_{21}+b_2\sigma_{22}+b_3\sigma_{23}} & e^{b_1\sigma_{31}+b_2\sigma_{32}+b_3\sigma_{33}} \end{pmatrix}\right\}$$

**{t1 = γ0 // dm[1, 2, 1] // ui,
t2 = γ0 // ui // dm[1, 2, 1], t1 == t2 // Simplify} // ColumnForm**

$$\left(-e^{b_1(\sigma_{11}+\sigma_{12}+\sigma_{21}+\sigma_{22}+\sigma_{31}+\sigma_{32})+b_3(\sigma_{13}+\sigma_{23}+\sigma_{33})}\,\omega[-b_1,-b_1,-b_3]\,(\alpha_{23}[-b_1,-b_1,-b_3]\,(-(-1+\alpha_{12}[-b_1,-b_1 \right.$$

$$\left(e^{b_1(\sigma_{11}+\sigma_{12}+\sigma_{21}+\sigma_{22}+\sigma_{31}+\sigma_{32})+b_3(\sigma_{13}+\sigma_{23}+\sigma_{33})}\,\omega[-b_1,-b_1,-b_3]\,(\alpha_{23}[-b_1,-b_1,-b_3]\,((-1+\alpha_{12}[-b_1,-b_1,- \right.$$

True

{t1 = γ0 // dS[1] // ui, t2 = γ0 // ui // dS[1], t1 == t2 // bSimplify} // ColumnForm

$$\left(-e^{-b_1(\sigma_{21}+\sigma_{31})+b_2(\sigma_{22}+\sigma_{32})+b_3(\sigma_{23}+\sigma_{33})}\,\omega[b_1,-b_2,-b_3]\,(\alpha_{23}[b_1,-b_2,-b_3]\,\alpha_{32}[b_1,-b_2,-b_3]-\alpha_{22}[b_1,- \right.$$

s₁

s₂

s₃

Γ

$$\left(-e^{-b_1(\sigma_{21}+\sigma_{31})+b_2(\sigma_{22}+\sigma_{32})+b_3(\sigma_{23}+\sigma_{33})}\,\omega[b_1,-b_2,-b_3]\,(\alpha_{23}[b_1,-b_2,-b_3]\,\alpha_{32}[b_1,-b_2,-b_3]-\alpha_{22}[b_1,- \right.$$

s₁

s₂

s₃

Γ

True

The twist equation.

Υ0 = Γ[Vi] // bSimplify

$$\begin{pmatrix} \frac{((-1+e^{b_1+b_2})\, b_1 b_2)^{1/4}}{((-1+e^{b_1})(-1+e^{b_2})(b_1+b_2))^{1/4}} & s_1 & s_2 \\ s_1 & \frac{e^{b_2}-e^{b_1+b_2}-e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1 (b_1+b_2)}}}{1-e^{b_1+b_2}} & \frac{e^{b_2}-e^{b_1+b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}{b_2 (b_1+b_2)}}}{1-e^{b_1+b_2}} \\ s_2 & \frac{1-e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1 (b_1+b_2)}}}{1-e^{b_1+b_2}} & -\frac{-1+e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_1}{b_2 (b_1+b_2)}}}{1-e^{b_1+b_2}} \\ \Gamma & 1 & e^{-\frac{b_1}{2}} \end{pmatrix}$$

$$\text{Series}\left[\text{PowerExpand}\left[\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1 (b_1+b_2)}}\ /.\ b_{i\_} \mapsto \hbar b_i\right], \{\hbar, 0, 5\}\right]$$

$$b_2 \hbar + \frac{1}{2} b_2 (b_1+b_2) \hbar^2 + \frac{1}{24} b_2 \left(4 b_1^2 + 7 b_1 b_2 + 4 b_2^2\right) \hbar^3 + \frac{1}{48} b_2 \left(2 b_1^3 + 5 b_1^2 b_2 + 5 b_1 b_2^2 + 2 b_2^3\right) \hbar^4 + \frac{b_2 \left(16 b_1^4 + 52 b_1^3 b_2 + 73 b_1^2 b_2^2 + 52 b_1 b_2^3 + 16 b_2^4\right) \hbar^5}{1920} + O[\hbar]^6$$

$$\text{Series}\left[\text{PowerExpand}\left[\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})}\ /.\ b_{i\_} \mapsto \hbar b_i\right], \{\hbar, 0, 5\}\right]$$

$$\sqrt{b_1}\sqrt{b_2}\sqrt{b_1+b_2}\,\hbar^{3/2} + \frac{1}{2}\left(b_1^{3/2}\sqrt{b_2}\sqrt{b_1+b_2} + \sqrt{b_1}\, b_2^{3/2}\sqrt{b_1+b_2}\right)\hbar^{5/2} + \frac{1}{24}\left(4 b_1^{5/2}\sqrt{b_2}\sqrt{b_1+b_2} + 7 b_1^{3/2} b_2^{3/2}\sqrt{b_1+b_2} + 4\sqrt{b_1}\, b_2^{5/2}\sqrt{b_1+b_2}\right)\hbar^{7/2} + \frac{1}{48}\left(2 b_1^{7/2}\sqrt{b_2}\sqrt{b_1+b_2} + 5 b_1^{5/2} b_2^{3/2}\sqrt{b_1+b_2} + 5 b_1^{3/2} b_2^{5/2}\sqrt{b_1+b_2} + 2\sqrt{b_1}\, b_2^{7/2}\sqrt{b_1+b_2}\right)\hbar^{9/2} + O[\hbar]^{11/2}$$

Υ0 ** (Υ0 // dA[1, 2])

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 0 \\ s_2 & 0 & 1 \\ \Gamma & 1 & 1 \end{pmatrix}$$

θ0 = Υ0 ** (Xp[1, 2] // Γ) ** (Υ0 // dS[1, 2] // dσ[1 → 2, 2 → 1])

$$\begin{pmatrix} 1 \\ s_1 \quad \frac{e^{-b_1}\left(-\frac{i\, e^{-\frac{b_2}{2}}\left(-1+e^{b_1}+e^{\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1(b_1+b_2)}}\right)\left(e^{\frac{b_1}{2}}\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{\cdots}}\right.}{\sqrt{-1+e^{-b_1-b_2}}\,(-1+e^{b_2})\cdots}\right.}{\cdots} \\ s_2 \quad \frac{e^{-\frac{b_1}{2}}\left(\frac{\left(-1+e^{b_1}+e^{\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})\, b_2}{b_1(b_1+b_2)}}\right)\left(e^{\frac{b_1}{2}}\sqrt{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2})}\, b_1-(-1+e^{b_2})\sqrt{b_1 b_2 (b_1+b_2)}\right)}{(-1+e^{b_2})(-1+e^{b_1+b_2})^{3/2}} - \frac{i\, e^{\frac{3 b_1}{2}-\frac{b_2}{2}}\left(1-e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1+e^{b_1})(-1+e^{b_2})(-1+e^{b_1+b_2}\cdots}{b_1(b_1+b_2)}}\right.\cdots}{\cdots}\right.}{\cdots} \\ \Gamma \end{pmatrix}$$

```
ϒ0 ** (Xp[1, 2] // Γ) ** (ϒ0 // dS[1, 2])
```

$$\frac{\left(\left(-1+e^{b_1+b_2}\right) b_1 b_2\right)^{1/4} \left(\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(b_1+b_2\right)}{\left(-1+e^{b_1+b_2}\right) b_1 b_2}\right)^{1/4}}{\left(\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(b_1+b_2\right)\right)^{1/4}}$$

$$s_1 \quad \frac{e^{-b_2}\left(\frac{\left(1-e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right) b_2}{b_1\left(b_1+b_2\right)}}\right)\left(-e^{\frac{b_1}{2}}\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)}\, b_1+\left(-1+e^{b_1}\right)\sqrt{b_1 b_2 (\ldots}\right.}{\sqrt{-1+e^{-b_1-b_2}}\left(1-e^{b_1+b_2}\right)}\right.}{\ldots}$$

$$s_2 \quad \frac{e^{-b_2}\left(1-e^{b_2}+e^{-\frac{b_1}{2}}\sqrt{\frac{(-1 \ldots}{}}\right.}{\ldots}$$

$$\Gamma$$

```
ϒ0 ** (Xp[1, 2] // Γ) ** (ϒ0 // dA[1, 2])
```

$$1$$

$$s_1 \quad \frac{-\frac{e^{\frac{b_1}{2}}\left(-e^{\frac{b_1}{2}}+e^{\frac{b_1}{2}+b_2}-\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right) b_2}{b_1\left(b_1+b_2\right)}}\right)\left(-\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)}\, b_1+e^{\frac{b_1}{2}+b_2}\left(-1+e^{b_1}\right)\sqrt{b_1 b_2\left(b_1+b_2\right)}\right)}{\left(-1+e^{b_1+b_2}\right)^{3/2}}+\frac{\left(-1+e^{b_1}+e^{\frac{b_1}{2}}\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right) b_2}{b_1\left(b_1+b_2\right)}}\right)\left(\ldots\right.}{\ldots}}{\sqrt{\left(-1+e^{b_1}\right)\ldots}}$$

$$s_2 \quad -\frac{\sqrt{\left(-1+e^{b_1}\right) b_2}\left(\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)(-1 \ldots}\right.}{\sqrt{\left(-1+e^{b_2}\right)\ldots}}$$

$$\Gamma$$

$$\frac{1}{\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)}\,(b_1+b_2)}$$

$$\left(-\left(\left(e^{\frac{b_1}{2}}\left(-e^{\frac{b_1}{2}}+e^{\frac{b_1}{2}+b_2}-\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}{b_1\,(b_1+b_2)}}\right)\right.\right.\right.$$

$$\left.\left(-\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)}\,b_1+e^{\frac{b_1}{2}+b_2}\left(-1+e^{b_1}\right)\sqrt{b_1\,b_2\,(b_1+b_2)}\right)\right)\Big/$$

$$\left.\left(-1+e^{b_1+b_2}\right)^{3/2}\right)+\left(\left(\left(-1+e^{b_1}+e^{\frac{b_1}{2}}\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}{b_1\,(b_1+b_2)}}\right)\right.\right.$$

$$\left(e^{b_1+b_2}\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1^2\sqrt{\left(-1+e^{b_1+b_2}\right)b_2}\;+\right.$$

$$\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1\sqrt{\left(-1+e^{b_1+b_2}\right)b_2}\left(-1+e^{b_1+b_2}+e^{b_1+b_2}\,b_2\right)+e^{\frac{b_1}{2}}\left(-1+e^{b_1+b_2}\right)$$

$$\left.\left.\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1\,(b_1+b_2)}\left(e^{b_2}\left(-1+e^{b_1}\right)b_1+\left(-1+e^{b_2}\right)b_2\right)\right)\right)\Big/$$

$$\left(\left(-1+e^{b_1+b_2}\right)^2\left(\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}-e^{\frac{b_1}{2}+b_2}\sqrt{b_1\,(b_1+b_2)}\;+\right.\right.$$

$$\left.\left.\left.e^{\frac{3b_1}{2}+b_2}\sqrt{b_1\,(b_1+b_2)}\right)\right)\right)\;//\;\mathbf{FullSimplify}$$

$$\frac{1}{\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)}\,(b_1+b_2)}\Bigg(\Bigg(e^{\frac{b_1}{2}}\left(-e^{\frac{b_1}{2}}\left(-1+e^{b_2}\right)+\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}{b_1\,(b_1+b_2)}}\right)$$

$$\left(-\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)}\,b_1+e^{\frac{b_1}{2}+b_2}\left(-1+e^{b_1}\right)\sqrt{b_1\,b_2\,(b_1+b_2)}\right)\Bigg)\Big/$$

$$\left(-1+e^{b_1+b_2}\right)^{3/2}+\Bigg(\Bigg(-1+e^{b_1}+e^{\frac{b_1}{2}}\sqrt{\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}{b_1\,(b_1+b_2)}}\Bigg)$$

$$\Big(e^{b_1+b_2}\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1^2\sqrt{\left(-1+e^{b_1+b_2}\right)b_2}\;+$$

$$\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1\sqrt{\left(-1+e^{b_1+b_2}\right)b_2}\left(-1+e^{b_1+b_2}\,(1+b_2)\right)+$$

$$e^{\frac{b_1}{2}}\left(-1+e^{b_1+b_2}\right)\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)b_1\,(b_1+b_2)}\left(-b_2+e^{b_2}\left(\left(-1+e^{b_1}\right)b_1+b_2\right)\right)\Big)\Bigg)\Big/$$

$$\left(\left(-1+e^{b_1+b_2}\right)^2\left(\sqrt{\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\left(-1+e^{b_1+b_2}\right)b_2}+e^{\frac{b_1}{2}+b_2}\left(-1+e^{b_1}\right)\sqrt{b_1\,(b_1+b_2)}\right)\right)\Bigg)$$

$$\left\{n=2;\ \Upsilon=\Gamma\left[\omega[b_1,b_2],\ \sum_{i=1}^{n}h_i\prod_{j=1}^{n}T_j^{\sigma_{10\,i+j}},\ \sum_{i=1}^{n}\sum_{j=1}^{n}t_i\,h_j\,\alpha_{10\,i+j}[b_1,b_2]\right]\ //\ \text{bSimplify}\right\}$$

$$\left\{\begin{pmatrix}\omega[b_1,b_2] & s_1 & s_2\\ s_1 & \alpha_{11}[b_1,b_2] & \alpha_{12}[b_1,b_2]\\ s_2 & \alpha_{21}[b_1,b_2] & \alpha_{22}[b_1,b_2]\\ \Gamma & e^{b_1\sigma_{11}+b_2\sigma_{12}} & e^{b_1\sigma_{21}+b_2\sigma_{22}}\end{pmatrix}\right\}$$

$$\theta=\Upsilon**\left(\text{Xp}[1,2]\ //\ \Gamma\right)**\left(\Upsilon\ //\ \text{dS}[1,2]\ //\ \text{d}\sigma[1\to 2,\,2\to 1]\right)$$

$$\Big(-e^{b_1(\sigma_{11}+\sigma_{21})+b_2(\sigma_{12}+\sigma_{22})}\,\omega[-b_1,-b_2]\,\omega[b_1,b_2]\,(\alpha_{12}[-b_1,-b_2]\,\alpha_{21}[-b_1,-b_2]-\alpha_{11}[-b_1,-b_2]\,\alpha_{22}[-\text{…}$$

$$s_1$$

$$s_2$$

$$\Gamma$$

```
bSimplify[θ == ui[θ]]
```

```
(ⅇ^b1 α12[-b1, -b2] α12[b1, b2] - ⅇ^b1 α21[-b1, -b2] α21[b1, b2] +
     α11[-b1, -b2] (α11[b1, b2] - (-1 + ⅇ^b1) α21[b1, b2]) - α12[b1, b2] α22[-b1, -b2] +
     ⅇ^b1 α12[b1, b2] α22[-b1, -b2] - α22[-b1, -b2] α22[b1, b2]) /
   (α12[-b1, -b2] α21[-b1, -b2] - α11[-b1, -b2] α22[-b1, -b2]) == 0 &&
 (b1 (ⅇ^b1 α11[-b1, -b2] α12[b1, b2] + α21[-b1, -b2] ((-1 + ⅇ^b1) α12[b1, b2] - α22[b1, b2])) -
     b2 (-ⅇ^b1 α21[-b1, -b2] α22[b1, b2] +
         α11[-b1, -b2] (α12[b1, b2] - (-1 + ⅇ^b1) α22[b1, b2]))) /
   (α12[-b1, -b2] α21[-b1, -b2] - α11[-b1, -b2] α22[-b1, -b2]) == 0 &&
 (b1 (α11[b1, b2] α12[-b1, -b2] - α21[b1, b2] ((-1 + ⅇ^b1) α12[-b1, -b2] + ⅇ^b1 α22[-b1, -b2])) -
     b2 (-α21[b1, b2] α22[-b1, -b2] +
         α11[b1, b2] (ⅇ^b1 α12[-b1, -b2] + (-1 + ⅇ^b1) α22[-b1, -b2]))) /
   (α12[-b1, -b2] α21[-b1, -b2] - α11[-b1, -b2] α22[-b1, -b2]) == 0 &&
 (-ⅇ^b1 α11[-b1, -b2] α11[b1, b2] + α11[b1, b2] α21[-b1, -b2] -
     ⅇ^b1 α11[b1, b2] α21[-b1, -b2] + α21[-b1, -b2] α21[b1, b2] +
     ⅇ^b1 α22[-b1, -b2] α22[b1, b2] - α12[-b1, -b2] (α12[b1, b2] - (-1 + ⅇ^b1) α22[b1, b2])) /
   (α12[-b1, -b2] α21[-b1, -b2] - α11[-b1, -b2] α22[-b1, -b2]) == 0
```

```
Assuming[α12[-b1, -b2] α21[-b1, -b2] - α11[-b1, -b2] α22[-b1, -b2] ≠ 0,
 bSimplify[θ == ui[θ]]]
```

```
α11[-b1, -b2] (α11[b1, b2] - (-1 + ⅇ^b1) α21[b1, b2]) +
    ⅇ^b1 α12[b1, b2] (α12[-b1, -b2] + α22[-b1, -b2]) ==
   ⅇ^b1 α21[-b1, -b2] α21[b1, b2] + α22[-b1, -b2] (α12[b1, b2] + α22[b1, b2]) &&
 b1 (ⅇ^b1 α11[-b1, -b2] α12[b1, b2] + α21[-b1, -b2] ((-1 + ⅇ^b1) α12[b1, b2] - α22[b1, b2])) ==
   b2 (-ⅇ^b1 α21[-b1, -b2] α22[b1, b2] + α11[-b1, -b2] (α12[b1, b2] - (-1 + ⅇ^b1) α22[b1, b2])) &&
 b1 (α11[b1, b2] α12[-b1, -b2] - α21[b1, b2] ((-1 + ⅇ^b1) α12[-b1, -b2] + ⅇ^b1 α22[-b1, -b2])) ==
   b2
    (-α21[b1, b2] α22[-b1, -b2] + α11[b1, b2] (ⅇ^b1 α12[-b1, -b2] + (-1 + ⅇ^b1) α22[-b1, -b2])) &&
 ⅇ^b1 α11[-b1, -b2] α11[b1, b2] + ⅇ^b1 α11[b1, b2] α21[-b1, -b2] +
    α12[-b1, -b2] (α12[b1, b2] - (-1 + ⅇ^b1) α22[b1, b2]) ==
   α11[b1, b2] α21[-b1, -b2] + α21[-b1, -b2] α21[b1, b2] + ⅇ^b1 α22[-b1, -b2] α22[b1, b2]
```