Severa on Quantization of Lie BiAlgebras and Moduli of Flat Connections

Monday, August 24, 2015 10:58 AM

How to produce Non-commutative algebras?

A, B: assoc. algebras in some BMC $\mathcal{D}$

$A\otimes B$ is an algebra by

$(AB)(AB) \longrightarrow (AB)$ by

this may be NC even if A, B are commutative.

Usually $\mathcal{D}$ is a Drinfeld category:

$\mathfrak{g}$: Lie alg. $\mathcal{D}_0$: $U\mathfrak{g}$-modules

$t\in(S^2\mathfrak{g})^{\mathfrak{g}}$ defines braiding via

$$X\otimes Y \xrightarrow{\sigma\circ(1+\frac{\epsilon}{2}t+\dots)} Y\otimes X \quad \text{at } \epsilon^2=0$$

associator extend it to "big" ϵ:

$$\beta=\sigma\circ\exp\left(\tfrac{1}{2}\hbar t\right)$$ but w/ new associativity

$$\gamma=\gamma_0\circ\Phi(\hbar t^{12}, \hbar t^{23})$$

where Φ is an associator in $K\langle\langle x, y\rangle\rangle$

$A\in\mathcal{D}$ assoc. algebra is a rep.

of $\mathfrak{g}$ along w/ $A\otimes A \to A$,
associative rel. Φ:

$$\begin{tikzcd}
(AA)A \arrow[rr, "\Phi"] \arrow[d, "m\otimes 1"] & & A(AA) \arrow[d, "1\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

Commutative assoc. algebras:
If $A \in \mathcal{D}_0$ is commutative assoc. s.t. it is commutative even w.r.t. $1+\frac{1}{2}\epsilon t$: namely  $=0$

then A is a commutative assoc. alg. in $\mathcal{D}$.
(Same for co-comm, co-assoc, co-alg...)

If $\mathcal{D}, \mathcal{C}$ are BMCs & $F:\mathcal{D}\to\mathcal{C}$ is a functor, lax monoidal:
$\forall X, Y \in \mathcal{D}$, have

$$F(X)F(Y) \longrightarrow F(XY)$$ [not nec. is...]

$$F(X)F(Y) \longrightarrow F(XY) \quad \left(\begin{matrix}\text{not}\\ \text{nec.}\\ \text{iso}\end{matrix}\right)$$

compatible w/ assoc, s.t.

[if the arrow goes the other way, "co-monoidal".]

If $A \in \mathcal{D}$ is an algebra, then $F(A)$ is an algebra by

$$F(A)\cdot F(A) \longrightarrow F(A\cdot A) \longrightarrow F(A)$$

If $\mathfrak{h} \subset \mathfrak{g}$ is Lie sub-alg co-isotropic:

$$S^2\mathfrak{g} \longrightarrow S^2(\mathfrak{g}/\mathfrak{h})$$
$$t \longrightarrow 0$$

Then $F: \mathcal{D} = \mathcal{U}(\ )\text{-mod} \longrightarrow \text{Vect}$ by

$$X \longrightarrow X^{\mathfrak{h}}$$

is a BMF (B. M. Functor)

& $F: \mathcal{D} \to \text{Vect}$ by

$$X \longrightarrow X_{\mathfrak{h}}$$

is B. Co-monoidal Functor.

If $\mathcal{D}_0, \mathcal{C}_0$ are infinitesimally braided

& $F: \mathcal{D}_0 \to \mathcal{C}_0$ is inf-braided,

then

$$F: \mathcal{D}_\hbar^{\Phi} \longrightarrow \mathcal{C}_\hbar^{\Phi} \text{ is BMF.}$$

Example. Suppose $A \in \mathcal{D}_0$

$$A = C^\infty(M) \circlearrowleft \mathfrak{g}$$

by action on M $\mathfrak{g} \circlearrowright M$

When is  $= 0$?

Iff $\rho(t) = 0$, iff the stabilizers are co-isotropic in $\mathfrak{g}$.

In particular, $M = G/H$ for big enough H is a fine example.

How to get a Hopf algebra?

From: BMC $\mathcal{D} \ni Q$ co-comm-coalg.
object

$\mathcal{D} \xrightarrow{F} \mathcal{C}$ $\quad$ $\mathcal{C}$ also BMC

F braided comonoidal functor

s.t. "F kill one Q":

$$\begin{tikzcd} F(X\otimes Q\otimes Y) \arrow[r] \arrow[rr, bend right, "\text{iso}"'] & F(XQQY) \arrow[r] & F(XQ)F(QY) \end{tikzcd}$$

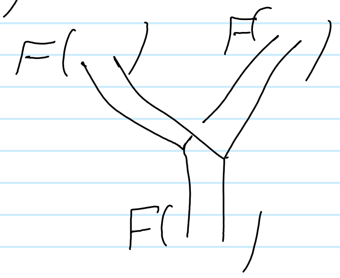
& also

$$\begin{tikzcd} F(Q) \arrow[r] \arrow[rr, bend right, "\text{iso}"'] & F(1_{\otimes}) \arrow[r] & 1_{\otimes} \end{tikzcd}$$

Then

$H := F(QQ)$ is a Hopf algebra

co-alg by:

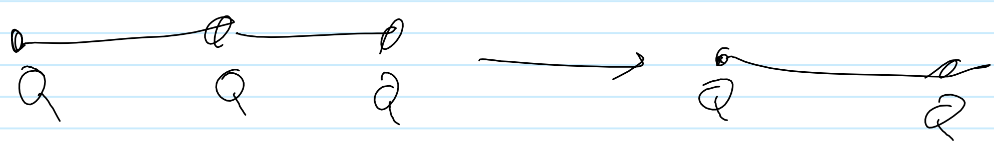


likewise for co-unit.

algebra by:

$$\begin{tikzcd} F(QQ)\otimes F(QQ) \arrow[r, dashed, "\text{new map}"] \arrow[d, "\cong"] & F(QQ) \\ F(QQQ) \arrow[ur, "F(1\in 1)"'] & \end{tikzcd}$$

in pictures:



unit:

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "\text{new}"', "\text{map}"] \arrow[d, phantom, "\cong" rotate=90] & F(QQ) \\ F(Q) \arrow[ur, "F(\Delta)"'] & \end{tikzcd}$$

antipode: $S: F(QQ) \to F(QQ)$ by braiding the two Q's: $F(\times)$

Proof of associativity:

$$\begin{tikzcd} F(QQ)F(QQ)F(QQ) \arrow[r, shift left, "LHS"] \arrow[r, shift right, "RHS"'] \arrow[d, phantom, "\cong" rotate=90] & F(QQ) \\ F(QQQQ) \arrow[ur, "F(1\epsilon\epsilon 1)"'] & \end{tikzcd}$$

G: group $\rightsquigarrow$ $B_\bullet G = E_\bullet G / G$

where

$$E_n G = G^{n+1}$$

48:00