Les Diablerets handout on 150818

Tuesday, August 18, 2015 9:07 AM

Booklet (?): These 4 pages, then 6-10 more:

1. GT (2 pages)
2. BF in Vienna+ (2 pages)
3. Pages 33 34, Burnt+ "FT of KBH" (2 pages)
4. 4 pages from Montpellier ??

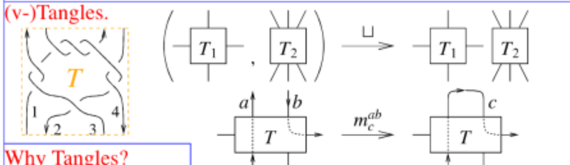
Dror Bar-Natan: Talks: LesDiablerets-1508:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

Work in Progress on

Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

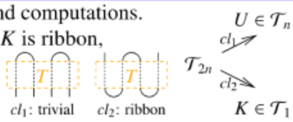
(v-)Tangles.



(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta) : cl_1(\zeta) = 1\}$.



(Genus and crossing number are also definable properties). (blackboard aside on genus) ?? Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a, T_b \to T_c,\ \mu := 1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $(|_a; {}_a\!\nearrow\!{}_b, {}_b\!\nwarrow\!{}_a) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array}; \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

stepping stones

Implementation key idea: ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ε, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ε; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t|h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + αδ/μ  ε + δθ/μ; ϕ + αψ/μ  Ξ + ψθ/μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1-T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1/T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h|t)_{a_} :> a, ∞];
  M = Outer[Factor[∂_{h_a, t_b}λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

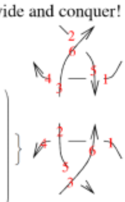
Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, ts}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hs}];
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
True
```

R3 ... divide and conquer!

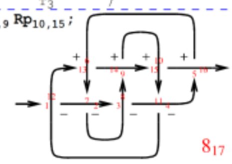
```
{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
 Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```

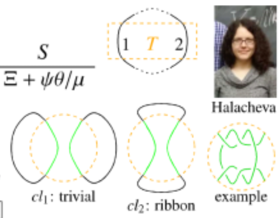
$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



Closed Components. The Halacheva trace tr_c satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi + \psi\theta/\mu\end{array}$$

```
tr_{c_}[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c}λ ∂_{t_c}λ; ∂_{h_c}λ λ) /. (t|h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ*θ/(1 - α)] // ΓCollect];
(ζ // m_{12→1} // tr1) == (ζ // m_{21→1} // tr1)
True
```

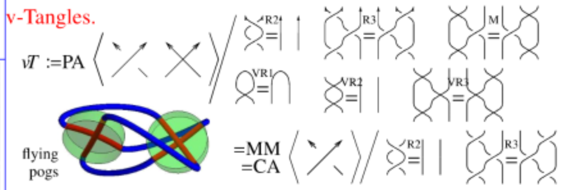


Halacheva

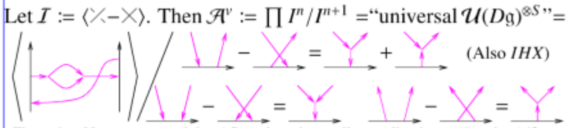
Weaknesses. • m^{ab}_c and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle\times - \times\rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=

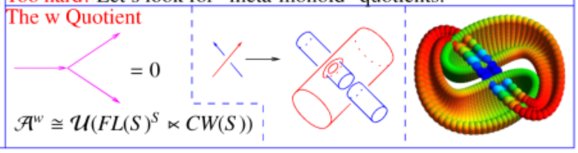


Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z: v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

A. Polyak & Ohtsuki @ Heian Shrine, Kyoto

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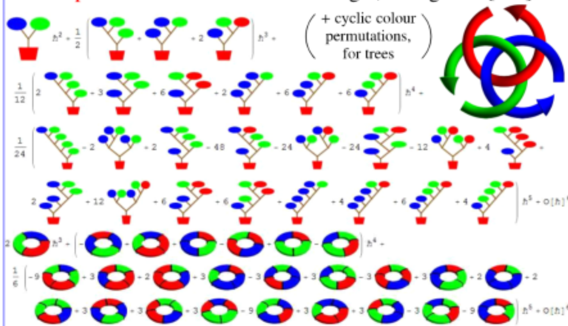
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

Work in Progress on **Polynomial Time Knot Polynomials, B**

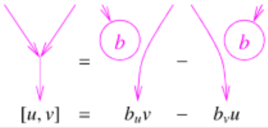
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:

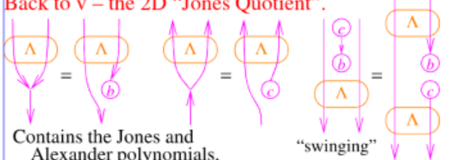


(I have a fancy free-Lie calculator!) (ωεβ/FLD) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



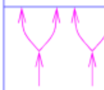
Back to v – the 2D "Jones Quotient".



V. Jones

Contains the Jones and Alexander polynomials, "swinging" . . . yet still too hard!

The OneCo Quotient. Likely related to [ADO]



= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

The Abstract Context

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

$$\text{meta-associativity:} \quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

$$\text{meta-locality:} \quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:} \quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with

$$\Gamma_1(S) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := R_S\langle S \rangle).$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

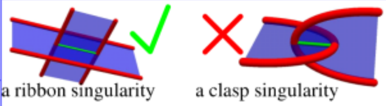
Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c\right)$$

What's done? The braid part, with still-ugly formulas.

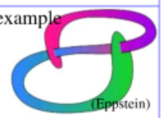
What's missing? A lot of concept- and detail-sensitive work towards m^{ab}_{1c}, α^{ab}, and ρ^{ab}_c. The "ribbon element".

B ✗



a ribbon singularity ✓ ✗ a clasp singularity

example



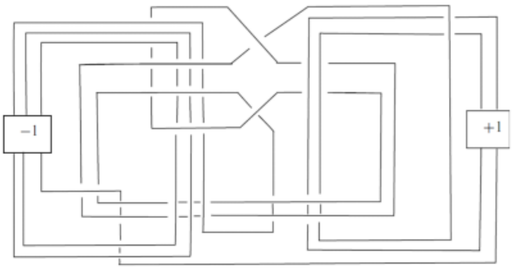
(Eppstein)

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

[GST]: a slice knot that might not be ribbon (48 crossings).



References.

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[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

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[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Help Needed!

I'm slow and feeble-minded.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

B. Snakes & ladders pic? ✗

Dror Bar-Natan: Talks: LesDiablerets-1508: FreeLie‘ Demo

ωεβ:=www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/
see especially ωεβ/FLD, ωεβ/WKO4, and ωεβ/PP.

Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages FreeLie‘ and AwCalculus‘ are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⟶, ⟵}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
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```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y},\ \frac{1}{24}\overline{x\overline{\overline{xy}y}},$$
$$-\frac{1}{720}\overline{x\overline{x\overline{x\overline{xy}}}}+\frac{1}{180}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{180}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{120}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},$$
$$-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1440}+\frac{1}{360}\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}+\frac{1}{240}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{1}{720}\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}-\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{1440},\ \ldots\Big]$$

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ħ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] -
      adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}}+\frac{1}{80}\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{\overline{xy}y}y},$$
$$-\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},$$
$$\frac{\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{5040}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1344}+\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{15120}+\frac{1}{840}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{3360}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{6720}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1260}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1680}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{10080},\ \ldots\Big],$$
$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}}+\frac{1}{120}\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\overline{\overline{\overline{xy}y}y},$$
$$-\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},$$
$$\frac{\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{10080}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{2016}+\frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1890}+\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{1120}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{5040}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{2520}+\frac{1}{840}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1260}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{5040},\ \ldots\Big]\Big\}$$

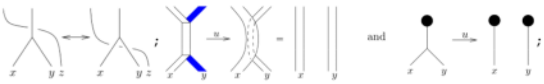
Meaningless calculations.

```
{b[F, G], tr_x[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{48}\overline{\overline{\overline{xy}y}y},\ \frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}-\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}-\frac{\overline{\overline{xy}\overline{\overline{xy}y}}}{1440}-\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{360}\overline{\overline{\overline{\overline{xy}y}y}y},$$
$$\frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1440}-\frac{1}{480}\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}-\frac{1}{288}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}-\frac{7\,\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{2880}+\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{2880},\ \ldots\Big],$$
$$\mathrm{CWS}\Big[-\frac{\overline{y}}{6},\ \frac{\overline{yy}}{24},\ \frac{\overline{xxy}}{180}+\frac{\overline{xyy}}{80}-\frac{\overline{yyy}}{360},\ -\frac{\overline{xxyy}}{180}+\frac{\overline{xyxy}}{240}-\frac{\overline{xyyy}}{240}-\frac{\overline{yyyy}}{1440},$$
$$-\frac{\overline{xxxxy}}{5040}+\frac{\overline{xxxyy}}{6720}-\frac{\overline{xxyxy}}{1120}+\frac{2\,\overline{xxyyy}}{945}-\frac{\overline{xyxyy}}{336}+\frac{\overline{xyyyy}}{6720}+\frac{\overline{yyyyy}}{10080},$$
$$\frac{\overline{xxxxyy}}{3360}-\frac{\overline{xxxyxy}}{1344}-\frac{\overline{xxxyyy}}{2240}+\frac{\overline{xxyxxy}}{2016}+\frac{13\,\overline{xxyxyy}}{10080}+\frac{\overline{xxyyxy}}{1680}-\frac{\overline{xxyyyy}}{3780}-\frac{\overline{xyxyxy}}{840}+\frac{\overline{xyxyyy}}{5040}+\frac{\overline{xyyxyy}}{2240}+\frac{\overline{xyyyyy}}{6720}+\frac{\overline{yyyyyy}}{60480},\ \ldots\Big]\Big\}$$

Also implemented: div, j, tb, DΔ, C^Δ, ...

The [BND] “vertex” equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ħ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

dε(x,y)

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>

$$\Big\{\mathrm{Es}\Big[\Big\langle\overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\overline{x\overline{xy}}}}{5760}-\frac{7\,\overline{x\overline{\overline{xy}y}}}{5760}+\frac{\overline{\overline{\overline{xy}y}y}}{1440},\ 0,$$
$$-\frac{31\,\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{967680}+\frac{31\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{83\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{967680}-\frac{31\,\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{725760}-\frac{31\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{645120}+$$
$$\frac{13\,\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{241920}+\frac{101\,\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1451520}+\frac{527\,\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{5806080}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{60480},\ \ldots\Big],$$
$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\overline{x\overline{xy}}}}{5760}-\frac{1}{720}\overline{x\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{\overline{xy}y}y},\ -\frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{7680}+\frac{\overline{x\overline{x\overline{\overline{xy}y}}}}{3840}-\frac{\overline{\overline{x\overline{xy}}\overline{xy}}}{6912},$$
$$-\frac{\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{645120}+\frac{23\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{161280}-\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{22680}-$$
$$\frac{41\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{580608}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{15120}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{12096}+\frac{71\,\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{483840}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{30240},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880}+\frac{\overline{xxyy}}{2880}+\frac{\overline{xyxy}}{5760}+\frac{\overline{xyyy}}{2880},\ 0,$$
$$-\frac{\overline{xxxxxy}}{120960}-\frac{\overline{xxxxyy}}{120960}-\frac{\overline{xxxyxy}}{120960}-\frac{\overline{xxxyyy}}{120960}-\frac{\overline{xxyxxy}}{241920}-\frac{\overline{xxyxyy}}{120960}-$$
$$\frac{\overline{xxyyxy}}{120960}-\frac{\overline{xxyyyy}}{120960}-\frac{\overline{xyxyxy}}{362880}-\frac{\overline{xyxyyy}}{120960}-\frac{\overline{xyyxyy}}{241920}-\frac{\overline{xyyyyy}}{120960},\ \ldots\Big]\Big],$$
$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11520},\ 0,\ -\frac{\overline{xxxxxx}}{725760},\ \ldots\Big]\Big\}$$

From V to F to KV following [AT].

```
logF = Λ[V][[1]] // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb]
```

```
⟨x → LS[y/2, xy/6, 1/24 xyy, -1/180 xxxy + 1/80 xxyy + 1/360 xyyy,
  -1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy -
  xyyyy/1440, xxxxxy/5040 - xxxxyy/1344 + 13 xxxyyy/15120 + 1/840 xxyxyy +
  xxxyxy/3360 + xxyyyy/6720 + xyxyyy/1260 + xxyyxy/1680 - xyyyyy/10080, ...],
 y → LS[0, xy/12, 1/24 xyy, -1/360 xxxy + 1/120 xxyy + 1/180 xyyy,
  -1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy -
  xyyyy/1440, xxxxxy/10080 - xxxxyy/2016 + xxxyyy/1890 + xxyxyy/1120 + xxxyxy/5040 +
  xxyyyy/2520 + 1/840 xyxyyy + xxyyxy/1260 - xyyyyy/5040, ...]⟩
```

V from Z_B, following [AET, BND].

```
(El[Z_B // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη^1 // tη^3 //
    hη^2 // hη^4 // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])[[
  1]]
```

clean

```
⟨1 → LS[0, -12/24, 0, 71112/5760 - 71122/5760 + 1222/1440, 0,
  -31111112/967680 + 3111122/483840 - 8311 1222/967680 - 311 12122/725760 - 311 11212/645120 +
  131 12222/241920 + 101 121222/1451520 + 527 112212/5806080 - 122222/60480, ...],
 2 → LS[1/2, -12/12, 0, 1112/5760 - 1/720 1122 + 1/720 1222,
  -11112/7680 + 11122/3840 - 11212/6912,
  -111112/645120 + 23 111122/483840 - 1311 1222/161280 - 112122/22680 - 411 11212/580608 +
  112222/15120 + 121222/12096 + 711 12212/483840 - 122222/30240, ...]⟩
```

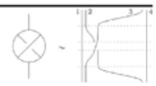
Solving for an associator Φ.

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ -Φ) ∧
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

```
DKS[0, 1/24 t13 t23, 0, -7 t13 t23 t23 t23/5760 + 7 t13 t13 t23 t23/5760 - t13 t13 t13 t23/1440,
 0, 31 t13 t23 t23 t23 t23 t23/967680 - 157 t13 t13 t23 t23 t13 t23/1935360 -
  31 t13 t23 t13 t23 t23 t23/387072 - 31 t13 t13 t23 t23 t23 t23/483840 +
  11 t13 t13 t13 t23 t13 t23/290304 + 31 t13 t13 t23 t13 t23 t23/725760 + 83 t13 t13 t13 t23 t23 t23/967680 -
  13 t13 t13 t13 t13 t23 t23/241920 + t13 t13 t13 t13 t13 t23/60480, ...]
```

The “buckle” Z_B, from Φ.



```
R = DKS[t[1, 2]/2];
Z_B = (-Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (-Φ)^σ[1,2,3] **
   Φ^σ[12,3,4];
Z_B@{4}
```

```
DKS[t23/2, -1/12 t13 t23 - 1/24 t14 t24 + 1/24 t14 t34 + 1/12 t24 t34,
 0, t13 t23 t23 t23/5760 + 7 t14 t24 t24 t24/5760 + t14 t34 t24 t24/1920 -
  t14 t34 t34 t24/1920 - 7 t14 t34 t34 t34/5760 - t24 t34 t34 t34/5760 + t14 t24 t34 t24/1920 +
  t14 t24 t14 t34/1920 - t14 t34 t24 t34/1920 - 1/720 t13 t13 t23 t23 +
  1/720 t13 t13 t13 t23 - 7 t14 t14 t24 t24/5760 + 7 t14 t14 t34 t34/5760 -
  t14 t24 t34 t34/5760 + t14 t14 t14 t24/1440 - t14 t14 t14 t34/1440 - 1/960 t14 t14 t24 t34 +
  t14 t24 t24 t34/5760 - 1/960 t24 t24 t34 t34 - t24 t24 t24 t34/5760, ...]
```

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ζ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```

```
Do[ζ = ζ // dm[r, k, r], {k, 1, 3}];
Do[ζ = ζ // dm[g, k, g], {k, 4, 6}];
Do[ζ = ζ // dm[b, k, b], {k, 7, 9}];
{ζ[[1]]_r@{5}, ζ[[2]]@{5}} // Print
```

```
{LS[0, bg, 1/2 bbg + bgr + 1/2 bgg,
  1/6 bbbg + 1/2 bbgr + 1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg,
  1/24 bbbbg + 1/6 bbbgr + 1/4 bbggr + 1/12 bbbgg +
  1/4 bbgrr + 1/6 bgggr + 1/4 bggrr - bbgrg +
  1/12 bbggg - 2 bbrgg + 1/6 bgrrr + 1/2 bgbgr -
  bgbrg - 1/12 bbgbg - 1/2 bgrgr + 1/24 bgggg, ...],
 CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, bbbgr/3 -
  bbgbr/2 + bbggr/2 + bbgrg/2 + bbgrr/2 + bbrbg/2 - 3 bbrgr/2 + bgbrr/2 - 3 bggbr/2 + bgggr/2 -
  bggrg/2 + bggrr/2 + bgrgg/2 - 3 bgrrg/2 + bgrrr/3 + brggr/2 - brgrr/2 + brrgr/2, ...]}
```

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yet×2.

Warning. Fidgety!