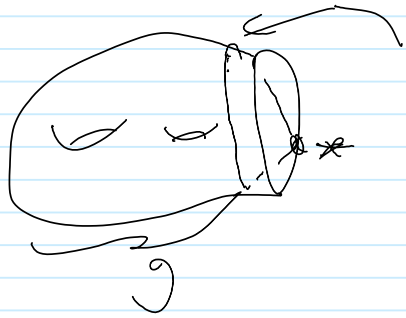
Kuno on Symplectic Expansions and Mapping Class Groups

Friday, August 21, 2015 10:28 AM

joint w/ Nariya Kawazumi

1. Background: Johnson Homomorphisms.

$\Sigma = \Sigma_{g,1} =$  {base loop}

$M_{g,1} = \pi_0(\mathrm{Diff}(\Sigma, \partial\Sigma))$

$\pi := \pi_1(\Sigma) = F_{2g}$

Dehn-Nielsen thm:

$M_{g,1} \cong \mathrm{Aut}_\zeta(\pi)$ {automorphisms preserving ζ}

Johnson's filtration:

$$\begin{array}{ccccc} M_{g,1}(0) & \supset M_{g,1}(1) & \supset & \cdots & M_{g,1}(k) \\ \| & \| & & & \\ M_{g,1} & \mathcal{I}_{g,1} & \text{: The Torelli group} & & \\ & \| & & & \\ & \mathrm{Ker}\,(M_{g,1} \to \mathrm{Aut}(H)) & & & \end{array}$$

$H = H_1(\Sigma, \mathbb{Q})$

$M_{g,1}(k) = \ker\left(M_{g,1} \longrightarrow \mathrm{Aut}\left(\begin{matrix}k\text{-th nilpotent}\\ \text{quotient of } \pi_1\end{matrix}\right)\right)$

$W = \sum_i A_iB_i - B_iA_i \in H^{\otimes 2}$ $\quad \{A_i, B_i\}$ symp. basis

$\hat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$: the completed tensor alg.

$\hat{\mathcal{L}} := \log Gr(\hat{T}) = \prod_{m=1}^{\infty} \mathcal{L}_m(H)$

Free Lie alg.

Johnson Homomorphism:

$\tau_k : M_{g,1}(k) \longrightarrow \mathrm{Der}^k_\omega(\hat{\mathcal{L}})$

deg. k derivations annihilating ω

in fact:

$\sum \tau_k = \tau : \bigoplus \frac{M_{g,1}(k)}{M_{g,1}(k+1)} \hookrightarrow \bigoplus \mathrm{Der}^k_\omega$

an embedding of Lie algebras.

Problem: Determine the co-kernel

Remark: Hain: Over $\mathbb{Q}$, $\operatorname{im}\tau$ is generated by its degree 1 part.

A geometric construction of τ_k

recall from morning, $\hat{\pi} = \pi/\text{conj.}$,

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}} \otimes \widehat{\mathbb{Q}\pi} \xrightarrow[\uparrow]{} \widehat{\mathbb{Q}\pi}$$

page of Kawazumi.

or:

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}} \longrightarrow \operatorname{Der}_\partial(\widehat{\mathbb{Q}\pi})$$

$$\begin{tikzcd}
T_{g,1} \arrow[r, hook, "DN"] \arrow[ddrrr, bend right, "\tau"'] & \operatorname{Aut}_\partial(\widehat{\mathbb{Q}\pi}) & & \operatorname{Der}_\partial(\widehat{\mathbb{Q}\pi}) \\
& \boxed{} \arrow[u, phantom, "\cup"] \arrow[rr, shift left, "\log"] & & \boxed{} \arrow[u, phantom, "\cup"] \arrow[ll, shift left, "\exp"] \\
& & & \widehat{\mathbb{Q}\hat{\pi}} \arrow[uu, bend right, "\sigma", "\cong"']
\end{tikzcd}$$

Example:

$C \subset \Sigma$ $\qquad$ $t_C \in \mathcal{M}_{g,1}$: Dehn twist on C

$x \in \pi$: a lift of C

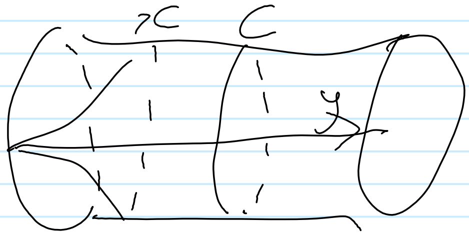
$$L(C) := \left|\tfrac{1}{2}(\log x)^2\right| \in \widehat{\mathbb{Q}\pi}$$

where $|\,| : \pi \to |\pi|$

Then (Kawazumi, K.) also Massuyeau-Turaev.

$$\tau(t_C) = L(C)$$

Essence of proof



$t_C(x) = x \quad t_C(y) = xy$

$$\rightsquigarrow \quad \log(DN(t_C))x = 0$$

$$\log \quad '' \quad y = (\log x)y$$