Pensieve header: One-Co computations in the abc presentation; continues pensieve://2015-07/. Jacobi holds and YB solved!

```
{ϵ1 = -1, ϵ2 = -1, ϵ6 = -1, ϵ7 = 1};
```

The bracket

On the elements *β*, a, c, *δ*a, ca, *δ*aa.

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];

Simp[expr_] := Simplify[expr];
S[β[f_]] := β[Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[δa[f_, i_, j_]] := δa[Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[δaa[f_, i_, j_, k_, l_]] := δaa[Simp[f], i, j, k, l];
S[expr_] := expr /. (λ_β | λ_a | λ_δa | λ_c | λ_ca | λ_δaa) ⧴ S[λ];

β[0] := 0;
β /: β[f_] + β[g_] := β[f + g] // S;
β /: g_ * β[f_] := β[g f] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f + g, j, k] // S;
a /: g_ * a[f_, j_, k_] := a[g f, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f + g, j] // S;
c /: g_ * c[f_, j_] := c[g f, j] // S;
δa[0, _, _] := 0;
δa /: δa[f_, j_, k_] + δa[g_, j_, k_] := δa[f + g, j, k] // S;
δa /: g_ * δa[f_, j_, k_] := δa[g f, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f + g, j, k, l] // S;
ca /: g_ * ca[f_, j_, k_, l_] := ca[g f, j, k, l] // S;
δaa[0, _, _, _, _] := 0;
δaa /: δaa[f_, i_, j_, k_, l_] + δaa[g_, i_, j_, k_, l_] :=
  δaa[f + g, i, j, k, l] // S;
δaa /: g_ * δaa[f_, i_, j_, k_, l_] := δaa[g f, i, j, k, l] // S;
```

δaa relations

```
VS = False;
```

“First sort tails then sort heads”

Standard Swinging - sorts heads, if support is 4 strands:

```
S[δaa[f_, i_, j_, k_, l_]] /; DQ[i, j, k, l] ∧ OQ[i, k] ∧ ! OQ[j, l] := (
   If[VS, Print["Standard swinging on ", δaa[f, i, j, k, l]]];
   S[δaa[f, i, l, k, j] + ca[b_k f, l, i, j] -
     ca[b_i f, l, k, j] - ca[b_k f, j, i, l] + ca[b_i f, j, k, l]]
  );
```

Locality - sorts tails when supports are disjoint:

```
S[δaa[f_, i_, j_, k_, l_]] /; ({i, j} ⋂ {k, l} === {}) ∧ ! OQ[i, k] := (
   If[VS, Print["Locality on ", δaa[f, i, j, k, l]]];
   δaa[f, k, l, i, j] // S
  );
```

Commute Heads - sorts tails when the heads are the same:

```
S[δaa[f_, i_, k_, j_, k_]] /; DQ[i, j, k] ∧ ! OQ[i, j] := (
   If[VS, Print["Commute heads on ", δaa[f, i, k, j, k]]];
   S[Expand[
     δaa[f, j, k, i, k] + ϵ2 (δa[b_i f, j, k] - δa[b_j f, i, k])
    ]]
  );
```

Commute Head/Tail - sorts tails:

```
S[δaa[f_, i_, j_, k_, i_]] /; DQ[i, j, k] ∧ ! OQ[i, k] := (
   If[VS, Print["Commute head/tail on ", δaa[f, i, j, k, i]]];
   S[
    δaa[f, k, i, i, j] + δaa[f, k, j, i, j] - δaa[f, i, j, k, j]
   ]
  );
```

Commute Head/Tail - sorts heads where heads & tails are both broken:

```
S[δaa[f_, k_, j_, j_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["Commute head/tail on ", δaa[f, k, j, j, i]]];
   S[
    δaa[f, j, i, k, j] + δaa[f, j, i, k, i] - δaa[f, k, i, j, i]
   ]
  );
```

2113 Swinging - sorts tails:

```
δaa[f, j, i, ii, k] // S
```

```
Locality on δaa[f, j, i, ii, k]

Standard swinging on δaa[f, ii, k, j, i]
```

$ca[-f\,b_{ii}, i, j, k] + ca[f\,b_{ii}, k, j, i] + ca[-f\,b_j, k, ii, i] + ca[f\,b_j, i, ii, k] + \delta aa[f, ii, i, j, k]$

```
S[δaa[f_, j_, i_, i_, k_]] /; DQ[i, j, k] ⋀ OQ[i, j, k] := (
   If[VS, Print["2113 swinging on ", δaa[f, j, i, i, k]]];
   S[ca[-f bi, i, j, k] + ca[f bi, k, j, i] +
     c[-f bj, k] ** aop[1, i] + ca[f bj, i, i, k] + aop[f, i] ** δa[1, j, k]]
  );
```

```
δaa[f, j, i, i, k] // S
```

$c[-f\,b_i\,b_j\,\epsilon_1, k] + ca[-f\,b_i, i, j, k] + ca[f\,b_i, k, j, i] + ca[-f\,b_j, k, i, i] + ca[f\,b_j, i, i, k] + \delta a[f\,b_i\,\epsilon_1, j, k] + \delta aa[f, i, i, j, k]$

3112 Swinging - sorts tails:

```
δaa[f, k, i, ii, j] // S
```

```
Locality on δaa[f, k, i, ii, j]

Standard swinging on δaa[f, ii, j, k, i]
```

$ca[-f\,b_{ii}, i, k, j] + ca[f\,b_{ii}, j, k, i] + ca[-f\,b_k, j, ii, i] + ca[f\,b_k, i, ii, j] + \delta aa[f, ii, i, k, j]$

```
S[δaa[f_, k_, i_, i_, j_]] /; DQ[i, j, k] ⋀ OQ[i, j, k] := (
   If[VS, Print["3112 swinging on ", δaa[f, k, i, i, j]]];
   S[ca[-f bi, i, k, j] + ca[f bi, j, k, i] +
     c[-f bk, j] ** aop[1, i] + ca[f bk, i, i, j] + aop[1, i] ** δa[f, k, j]]
  );
```

```
δaa[f, k, i, i, j] // S
```

```
3112 swinging on δaa[f, k, i, i, j]
```

$c[-f\,b_i\,b_k\,\epsilon_1, j] + ca[-f\,b_i, i, k, j] + ca[f\,b_i, j, k, i] + ca[-f\,b_k, j, i, i] + ca[f\,b_k, i, i, j] + \delta a[f\,b_i\,\epsilon_1, k, j] + \delta aa[f, i, i, k, j]$

Tails Commute - sorts heads when the tails are the same:

```
S[δaa[f_, i_, j_, i_, l_]] /; DQ[i, j, l] ⋀ ! OQ[j, l] := (
   If[VS, Print["Tails commute on ", δaa[f, i, j, i, l]]];
   δaa[f, i, l, i, j] // S
  );
```

1321 Swinging - sorts heads:

```
δaa[f, i, k, j, ii] // S
```

```
Standard swinging on δaa[f, i, k, j, ii]
```

```
ca[-f b_i, ii, j, k] + ca[f b_i, k, j, ii] +
 ca[-f b_j, k, i, ii] + ca[f b_j, ii, i, k] + δaa[f, i, ii, j, k]
```

```
S[δaa[f_, i_, k_, j_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1321 swinging on ", δaa[f, i, k, j, i]]];
   S[ca[-f b_i, i, j, k] + ca[f b_i, k, j, i] +
     ca[-f b_j, k, i, i] + ac[f b_j, i, k, i] + δaa[f, i, i, j, k]]
  );
```

1322 Swinging - sorts heads, but breaks tails:

```
δaa[f, i, k, j, jj] // S
```

```
Standard swinging on δaa[f, i, k, j, jj]
```

```
ca[-f b_i, jj, j, k] + ca[f b_i, k, j, jj] +
 ca[-f b_j, k, i, jj] + ca[f b_j, jj, i, k] + δaa[f, i, jj, j, k]
```

```
S[δaa[f_, i_, k_, j_, j_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1322 swinging on ", δaa[f, i, k, j, j]]];
   S[ac[-f b_i, j, k, j] + ca[f b_i, k, j, j] +
     ca[-f b_j, k, i, j] + ca[f b_j, j, i, k] + δaa[f, j, k, i, j]]
  );
```

1332 Swinging - sorts heads:

```
δaa[f, i, k, kk, j] // S
```

```
Standard swinging on δaa[f, i, k, kk, j]
```

```
ca[-f b_i, j, kk, k] + ca[f b_i, k, kk, j] +
 ca[-f b_kk, k, i, j] + ca[f b_kk, j, i, k] + δaa[f, i, j, kk, k]
```

```
S[δaa[f_, i_, k_, k_, j_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1332 swinging on ", δaa[f, i, k, k, j]]];
   S[c[-f b_i, j] ** aop[1, k] + ca[f b_i, k, k, j] +
     ca[-f b_k, k, i, j] + ca[f b_k, j, i, k] + δa[f, i, j] ** aop[1, k]]
  );
```

```
δaa[f, i, k, k, j] // S
```

```
1332 swinging on δaa[f, i, k, k, j]
```

```
ca[-f b_i, j, k, k] + ca[f b_i, k, k, j] + ca[-f b_k, k, i, j] +
 ca[f b_k, j, i, k] + δa[f b_k ∈_1, i, j] + δaa[f, i, j, k, k]
```

1231 Swinging - sorts heads:

```
δaa[f, i, j, k, ii] // S

Standard swinging on δaa[f, i, j, k, ii]

ca[-f b_i, ii, k, j] + ca[f b_i, j, k, ii] +
 ca[-f b_k, j, i, ii] + ca[f b_k, ii, i, j] + δaa[f, i, ii, k, j]

S[δaa[f_, i_, j_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1231 swinging on ", δaa[f, i, j, k, i]]];
   S[ac[-f b_i, k, j, i] + ca[f b_i, j, k, i] +
     ca[-f b_k, j, i, i] + ac[f b_k, i, j, i] + δaa[f, i, i, k, j]]
  );
```

1211 sliding - sorts heads:

```
S[δaa[f_, i_, j_, i_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["1211 sliding on ", δaa[f, i, j, i, i]]];
   S[δaa[f, i, i, i, j]]
  );
```

2111 sliding - sorts tails:

```
S[δaa[f_, j_, i_, i_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2111 sliding on ", δaa[f, j, i, i, i]]];
   S[δaa[f, i, i, j, i]]
  );
```

2212 sliding - sorts tails:

```
S[δaa[f_, j_, j_, i_, j_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2212 sliding on ", δaa[f, j, j, i, j]]];
   S[δaa[f, i, j, j, j]]
  );
```

2221 sliding - sorts heads:

```
S[δaa[f_, j_, j_, j_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2221 sliding on ", δaa[f, j, j, j, i]]];
   S[δaa[f, j, i, j, j]]
  );
```

2231 Swinging - sorts heads:

```
δaa[f, j, jj, k, i] // S

Standard swinging on δaa[f, j, jj, k, i]

ca[-f b_j, i, k, jj] + ca[f b_j, jj, k, i] +
 ca[-f b_k, jj, j, i] + ca[f b_k, i, j, jj] + δaa[f, j, i, k, jj]
```

```
S[δaa[f_, j_, j_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["2231 swinging on ", δaa[f, j, j, k, i]]];
   S[ca[-f b_j, i, k, j] + ca[f b_j, j, k, i] +
     ac[-f b_k, j, i, j] + ca[f b_k, i, j, j] + δaa[f, j, i, k, j]]
  );
```

2331 Swinging - sorts heads:

```
δaa[f, j, k, kk, i] // S
```

```
ca[-f b_j, i, kk, k] + ca[f b_j, k, kk, i] +
 ca[-f b_kk, k, j, i] + ca[f b_kk, i, j, k] + δaa[f, j, i, kk, k]
```

```
S[δaa[f_, j_, k_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["2331 swinging on ", δaa[f, j, k, k, i]]];
   S[c[-f b_j, i] ** aop[1, k] + ca[f b_j, k, k, i] +
     ca[-f b_k, k, j, i] + ca[f b_k, i, j, k] + δa[f, j, i] ** aop[1, k]]
  );
```

Backie jkkj Swinging - sorts tails or heads:

```
δaa[f, j, k, kk, jj] // S
```

```
ca[-f b_j, jj, kk, k] + ca[f b_j, k, kk, jj] +
 ca[-f b_kk, k, j, jj] + ca[f b_kk, jj, j, k] + δaa[f, j, jj, kk, k]
```

```
S[δaa[f_, j_, k_, k_, j_]] /; DQ[j, k] := (
   If[VS, Print["Backie swinging on ", δaa[f, j, k, k, j]]];
   S[c[-f b_j, j] ** aop[1, k] + ca[f b_j, k, k, j] +
     ca[-f b_k, k, j, j] + ac[f b_k, j, k, j] + δa[f, j, j] ** aop[1, k]]
  );
```

NonCommutativeMultiply

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_, 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;
```

```
β[f_] ** a[g_, j_, k_] := a[f g, j, k];
β[f_] ** c[g_, j_] := c[f g, j];
c[g_, j_] ** β[f_] := c[f g, j];
β[f_] ** δa[g_, j_, k_] := δa[f g, j, k];
β[f_] ** ca[g_, i_, j_, k_] := ca[f g, i, j, k];
ca[g_, i_, j_, k_] ** β[f_] := ca[f g, i, j, k];
β[f_] ** δaa[g_, i_, j_, k_, l_] := δaa[f g, i, j, k, l];
δa[g_, j_, k_] ** β[f_] := δa[f g, j, k];
δ ** a[f_, i_, j_] := δa[f, i, j];
c[f_, i_] ** a[g_, j_, k_] := ca[f g, i, j, k];
a[f_, i_, j_] ** δa[g_, k_, l_] := δaa[f g, i, j, k, l];
δa[f_, i_, j_] ** a[g_, k_, l_] := δaa[f g, i, j, k, l];
```

```
δ ** _c = 0;
δ ** _δa = 0;
δ ** _ca = 0;
δ ** _δaa = 0;
_c ** _c = 0;
_c ** _δa = _δa ** _c = 0;
_c ** _ca = _ca ** _c = 0;
_c ** _δaa = _δaa ** _c = 0;
_δa ** _δa = 0;
_δa ** _δaa = _δaa ** _δa = 0;
_δa ** _ca = _ca ** _δa = 0;
```

```
NonCommutativeMultiply::ndef =
 "NonCommuatativeMultiply is not defined on {`1`,`2`}."
NonCommutativeMultiply[x_, y_] :=
  (Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
```

```
NonCommuatativeMultiply is not defined on {`1`,`2`}.
```

Bracket Generalities

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

The γ shortcuts

```
γ[f_, j_, k_] := δa[f, j, k] - c[b_j f, k] // S;
γ[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;
ac[f_, j_, k_, l_] := ca[f, l, j, k] + B[a[1, j, k], c[f, l]];
aop[f_, j_] := a[f, j, j] + β[ϵ1 f b_j] + c[ϵ1 f, j];
```

Fundamental Brackets

a-β, a-c, a-a, AS

```
B[a[j_, k_], β[g_]] := γ[∂_{b_j} g - ∂_{b_k} g, j, k];
B[β[g_], a[j_, k_]] := -B[a[j, k], β[g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k} ⋂ {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // S;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] - γ[1, j, k, l] // S;
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
(* backie *) B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[b_j, k, j] - a[b_k, j, k] + a[b_j, k, k] - a[b_k, j, j] + ca[1, k, k, j] -
   ca[1, j, j, k] + ca[1, k, j, j] - ca[1, j, k, k] + γ[1, j, k] - γ[1, k, j];
(* [tail, selfie] *) B[a[j_, k_], a[j_, j_]] /; DQ[j, k] := γ[-ϵ6, j, k] // S;
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
(* [head, selfie] *) B[a[j_, k_], a[k_, k_]] /; DQ[j, k] := γ[-ϵ7, j, k] // S;
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[k, k]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] := γ[-f g, j, k];
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] := γ[f g, j, k];
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k} ⋂ {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];
```

Vanishing brackets

```
B[_β, _β | δ | _c | _δa | _ca | _δaa] = 0;
B[_β | δ | _c | _δa | _ca | _δaa, _β] = 0;
B[δ | _c | _δa | _ca | _δaa, δ | _c | _δa | _ca | _δaa] = 0;
```

Composite Brackets

```
B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[1, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[1, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[1, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[1, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];
```

```
B::ndef = "B is not defined on {`1`,`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
```

```
B is not defined on {`1`,`2`}.
```

Testing Jacobi and Anti-Symmetry

```
FormalBasis[S_List, f_] := Module[{ff, n = Length@S, i, j, k, l},
   ff = f @@ Table[b_S[[i]], {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, S[[i]], S[[j]]], {i, n}, {j, n}],
      Table[c[ff, S[[i]]], {i, n}],
      Table[δa[ff, S[[i]], S[[j]]], {i, n}, {j, n}],
      Table[ca[ff, S[[i]], S[[j]], S[[k]]], {i, n}, {j, n}, {k, n}],
      Table[δaa[ff, S[[i]], S[[j]], S[[k]], S[[l]]], {i, n}, {j, n}, {k, i, n}, {l, j, n}]
     } /. 1[___] → 1
  ];
FormalBasis[n_Integer, f_] := FormalBasis[Range[n], f];
FormalPlusBasis[n_, f_] := Module[{ff},
   ff = f @@ Table[b_i, {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[c[ff, i], {i, n}],
      Table[δa[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[ca[ff, i, j, k], {i, n}, {j, n - 1}, {k, j + 1, n}],
      Table[δaa[ff, i, j, k, l], {i, n - 1}, {j, i + 1, n}, {k, n - 1}, {l, k + 1, n}]
     } /. 1[___] → 1
  ];

VS = False;
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalPlusBasis[3, f],
   FormalPlusBasis[3, g]
  ]], 0]
```

```
{}
```

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalBasis[3, f],
   FormalBasis[3, g]
  ]], 0]
```

{}

```
Jacobi[x1_, x2_, x3_] := Module[{Jac},
   Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
   If[Jac === 0, Jac, {x1, x2, x3} → Jac]
  ];
```

```
JacPlusErrors = DeleteCases[
  bas1 = FormalPlusBasis[4, f];
  bas2 = FormalPlusBasis[4, g];
  bas3 = FormalPlusBasis[4, h];
  Flatten[
   Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]]],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

{}

```
JacPlusErrors // Length
```

0

```
VS = False;
JacErrors = DeleteCases[
  bas1 = FormalBasis[4, f];
  bas2 = FormalBasis[4, g];
  bas3 = FormalBasis[4, h];
  Flatten[
   Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]]],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

{}

The Adjoint action

AutoAd

```
AutoAd[x_][y_] :=
  Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
   pows = NestList[B[x, #] &, y, 20];
   Print["pows computed for ", {x, y}, "..."];
   states = Union[
     Cases[pows, s_β | s_a | s_c | s_δa | s_ca | s_δaa :→ ReplacePart[s, 1 → _], ∞]];
   Sum[
    seq = Cases[{#}, states[[i]], ∞] & /@ pows;
    seq = Replace[seq, {{_[f_, ___]} :→ f, {} → 0}, {1}];
    Print["seq computed... ", states[[i]], " is ", i, "/", Length@states];
    dseq = Drop[seq, sh];
    If[Union[Length[MonomialList[#]] & /@ dseq] === {1} ∧
      Union[Length[FactorTermsList[#]] & /@ dseq] === {2},
     sf1 = FindSequenceFunction[FactorTermsList[#][[1]] & /@ dseq];
     sf2 = FindSequenceFunction[FactorTermsList[#][[2]] & /@ dseq];
     Print["sf1: ", sf1, " sf2: ", sf2];
     sf = (sf1[#] sf2[#] &),
     (*Else*) sf = FindSequenceFunction[dseq,
       FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
         "Polynomial", "RationalFunction", "HypergeometricTerm"}];
     Print["sf: ", sf];
    ];
    ReplacePart[states[[i]], 1 → Simplify[Sum[seq[[n + 1]]/n!, {n, 0, sh - 1}] + Sum[sf[n + 1 - sh]/n!, {n, sh, ∞}]]],
    {i, Length@states}]
  ];
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)
```

```
AutoAd[a[t, j, k]][a[1, k, j]]
```

pows computed for {a[t, j, k], a[1, k, j]}...

seq computed... a[_, j, j] is 1/20

sf1: -1 & sf2: $t^4 b_j^3 (t b_j)^{\#1} b_k$ &

seq computed... a[_, j, k] is 2/20

sf1: -1 & sf2: $t^4 b_j^3 (t b_j)^{\#1} b_k$ &

seq computed... a[_, k, j] is 3/20

sf1: 1 & sf2: $t^4 b_j^4 (t b_j)^{\#1}$ &

seq computed... a[_, k, k] is 4/20

sf1: 1 & sf2: $t^4 b_j^4 (t b_j)^{\#1}$ &

seq computed... c[_, j] is 5/20

sf1: $7 + 2\,\#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1} b_k$ &

seq computed... c[_, k] is 6/20

sf: $t^4 b_j^3 \left(b_j (-t b_j)^{\#1} + (-t b_j)^{\#1} b_k + 7 (t b_j)^{\#1} b_k + 2\,\#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... ca[_, j, j, k] is 7/20

sf: $-t^4 b_j^2 (t b_j)^{\#1} (b_j - 6 b_k - 2\,\#1 b_k)$ &

seq computed... ca[_, j, k, k] is 8/20

sf1: $-7 - 2\,\#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

seq computed... ca[_, k, j, j] is 9/20

sf: $-t^4 b_j^2 \left(b_j (-t b_j)^{\#1} + (-t b_j)^{\#1} b_k + 4 (t b_j)^{\#1} b_k + \#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... ca[_, k, j, k] is 10/20

sf: $-t^4 b_j^2 \left(b_j (-t b_j)^{\#1} + b_j (t b_j)^{\#1} + (-t b_j)^{\#1} b_k - 2 (t b_j)^{\#1} b_k - \#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... ca[_, k, k, j] is 11/20

sf1: $4 + \#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

seq computed... ca[_, k, k, k] is 12/20

sf1: $-3 - \#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

seq computed... δa[_, j, j] is 13/20

sf1: $-3 - \#1$ & sf2: $t^4 b_j^2 (t b_j)^{\#1} b_k$ &

seq computed... δa[_, j, k] is 14/20

sf: $-t^4 b_j^2 \left(b_j (-t b_j)^{\#1} + (-t b_j)^{\#1} b_k + 4 (t b_j)^{\#1} b_k + \#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... δa[_, k, j] is 15/20

sf1: $-4 - \#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

seq computed... δa[_, k, k] is 16/20

sf1: $-3 - \#1$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

seq computed... δaa[_, j, j, j, k] is 17/20

sf: $t^4 b_j \left(b_j (-t b_j)^{\#1} + b_j (t b_j)^{\#1} + (-t b_j)^{\#1} b_k - 5 (t b_j)^{\#1} b_k - 2\,\#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... δaa[_, j, j, k, k] is 18/20

sf1: $2 (3 + \#1)$ & sf2: $t^4 b_j^2 (t b_j)^{\#1}$ &

seq computed... δaa[_, j, k, j, k] is 19/20

sf: $t^4 b_j \left(b_j (-t b_j)^{\#1} + b_j (t b_j)^{\#1} + (-t b_j)^{\#1} b_k - 5 (t b_j)^{\#1} b_k - 2\,\#1 (t b_j)^{\#1} b_k\right)$ &

seq computed... δaa[_, j, k, k, k] is 20/20

sf1: $2 (3 + \#1)$ & sf2: $t^4 b_j^2 (t b_j)^{\#1}$ &

$$a\left[e^{t b_j}, k, j\right] + a\left[-1 + e^{t b_j}, k, k\right] + a\left[-\frac{\left(-1 + e^{t b_j}\right) b_k}{b_j}, j, j\right] + a\left[-\frac{\left(-1 + e^{t b_j}\right) b_k}{b_j}, j, k\right] +$$
$$c\left[\frac{\left(1 - e^{t b_j} + 2 e^{t b_j} t b_j\right) b_k}{b_j}, j\right] + c\left[-1 + e^{-t b_j} + \frac{e^{-t b_j}\left(1 - e^{2 t b_j} + 2 e^{2 t b_j} t b_j\right) b_k}{b_j}, k\right] +$$
$$ca\left[e^{t b_j} t, k, k, j\right] + ca\left[\frac{-1 + e^{t b_j} - 2 e^{t b_j} t b_j}{b_j}, j, k, k\right] + ca\left[\frac{-1 + e^{t b_j} - e^{t b_j} t b_j}{b_j}, k, k, k\right] +$$
$$ca\left[\frac{-2\left(-1 + e^{t b_j}\right) b_k + b_j\left(1 - e^{t b_j} + 2 e^{t b_j} t b_k\right)}{b_j^2}, j, j, k\right] +$$
$$ca\left[\frac{e^{-t b_j}\left(\left(-1 + e^{t b_j}\right) b_k + b_j\left(-1 + e^{t b_j} - e^{2 t b_j} t b_k\right)\right)}{b_j^2}, k, j, j\right] +$$
$$ca\left[\frac{1}{b_j^2} e^{-t b_j}\left(-\left(1 - 3 e^{t b_j} + 2 e^{2 t b_j}\right) b_k + b_j\left(-\left(-1 + e^{t b_j}\right)^2 + e^{2 t b_j} t b_k\right)\right), k, j, k\right] +$$
$$\delta a\left[-e^{t b_j} t, k, j\right] + \delta a\left[\frac{-1 + e^{t b_j} - e^{t b_j} t b_j}{b_j}, k, k\right] + \delta a\left[-\frac{\left(1 - e^{t b_j} + e^{t b_j} t b_j\right) b_k}{b_j^2}, j, j\right] +$$
$$\delta a\left[\frac{e^{-t b_j}\left(\left(-1 + e^{t b_j}\right) b_k + b_j\left(-1 + e^{t b_j} - e^{2 t b_j} t b_k\right)\right)}{b_j^2}, j, k\right] +$$
$$\delta aa\left[\frac{2 - 2 e^{t b_j} + 2 e^{t b_j} t b_j}{b_j^2}, j, j, k, k\right] + \delta aa\left[\frac{2 - 2 e^{t b_j} + 2 e^{t b_j} t b_j}{b_j^2}, j, k, k, k\right] +$$
$$\delta aa\left[\frac{1}{b_j^3} e^{-t b_j}\left(\left(1 - 4 e^{t b_j} + 3 e^{2 t b_j}\right) b_k + b_j\left(\left(-1 + e^{t b_j}\right)^2 - 2 e^{2 t b_j} t b_k\right)\right), j, j, j, k\right] +$$
$$\delta aa\left[\frac{1}{b_j^3} e^{-t b_j}\left(\left(1 - 4 e^{t b_j} + 3 e^{2 t b_j}\right) b_k + b_j\left(\left(-1 + e^{t b_j}\right)^2 - 2 e^{2 t b_j} t b_k\right)\right), j, k, j, k\right]$$

Ad

```
Ad[a[t_, j_, k_]][β[f_]] /; FreeQ[t, b_] :=
  β[f] + c[(1 - e^(-t b_j)) (∂_{b_k} f - ∂_{b_j} f), k] + δa[((e^(-t b_j) - 1) (∂_{b_k} f - ∂_{b_j} f))/b_j, j, k];
Ad[a[t_, j_, k_]][a[1, j_, k_]] /; FreeQ[t, b_] := a[1, j, k];
Ad[a[t_, j_, k_]][a[1, n_, i_]] /;
    FreeQ[t, b_] ∧ ({j, k} ∩ {n, i} === {}) := a[1, n, i];
Ad[a[t_, j_, k_]][a[1, j_, j_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  a[1, j, j] + c[-1 + e^(-t b_j), k] + δa[(1 - e^(-t b_j))/b_j, j, k];
Ad[a[t_, j_, k_]][a[1, k_, k_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  a[1, k, k] + c[1 - e^(-t b_j), k] + δa[(-1 + e^(-t b_j))/b_j, j, k];
Ad[a[t_, j_, k_]][a[1, i_, j_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
  a[1, i, j] + a[1 - e^(-t b_j), i, k] + a[((e^(-t b_j) - 1) b_i)/b_j, j, k] + ca[(1 - e^(-t b_j))/b_j, k, i, j] +
```

$$\mathrm{ca}\left[\frac{e^{-t\,b_j}-1}{b_j}, j, i, k\right] + \mathrm{ca}\left[\frac{b_i\left(1-e^{-t\,b_j}-t\,b_j\right)}{b_j^2}, j, j, k\right] + \mathrm{ca}\left[\frac{e^{-2\,t\,b_j}\,b_i\left(1-e^{t\,b_j}-e^{t\,b_j}\left(e^{t\,b_j}-2\right)t\,b_j\right)}{b_j^2}, k, j, k\right] + \mathrm{ca}\left[\frac{e^{-2\,t\,b_j}\left(e^{t\,b_j}\left(1-t\,b_j\right)-1\right)}{b_j}, k, i, k\right] + \delta\mathrm{a}\left[\frac{b_i\left(-1+e^{-t\,b_j}+t\,b_j\right)\epsilon_2}{b_j^2} + \frac{b_i\left(1-e^{-2\,t\,b_j}+\left(-1-e^{-t\,b_j}\right)t\,b_j\right)\epsilon_2}{b_j^2}, j, k\right] + \delta\mathrm{a}\left[-\frac{\left(-1+e^{-t\,b_j}+t\,b_j\right)\epsilon_2}{b_j} - \frac{\left(1-e^{-2\,t\,b_j}+\left(-1-e^{-t\,b_j}\right)t\,b_j\right)\epsilon_2}{b_j}, i, k\right] + \delta\mathrm{aa}\left[\frac{2\,e^{-t\,b_j}\,b_i\,(\mathrm{Sinh}[t\,b_j]-t\,b_j)}{b_j^3}, j, k, j, k\right] + \delta\mathrm{aa}\left[\frac{-1+e^{-t\,b_j}+t\,b_j}{b_j^2}, i, j, j, k\right] + \delta\mathrm{aa}\left[-\frac{1-e^{-2\,t\,b_j}+\left(-1-e^{-t\,b_j}\right)t\,b_j}{b_j^2}, i, k, j, k\right];$$

```
Ad[a[t_, j_, k_]][a[1, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
```

$$\mathrm{a}\left[e^{-t\,b_j}, i, k\right] + \mathrm{a}\left[\frac{\left(1-e^{-t\,b_j}\right)b_i}{b_j}, j, k\right] + \mathrm{ca}\left[\frac{2\,e^{-t\,b_j}\,b_i\,(\mathrm{Sinh}[t\,b_j]-t\,b_j)}{b_j^2}, k, j, k\right] + \mathrm{ca}\left[\frac{e^{-2\,t\,b_j}\left(1+e^{t\,b_j}\left(-1+t\,b_j\right)\right)}{b_j}, k, i, k\right] + \delta\mathrm{a}\left[-\frac{e^{-2\,t\,b_j}\,b_i\left(-1+e^{t\,b_j}\left(1-t\,b_j\right)\right)\epsilon_2}{b_j^2}, j, k\right] + \delta\mathrm{a}\left[\frac{e^{-2\,t\,b_j}\left(-1+e^{t\,b_j}\left(1-t\,b_j\right)\right)\epsilon_2}{b_j}, i, k\right] + \delta\mathrm{aa}\left[\frac{2\,e^{-t\,b_j}\,b_i\,(-\mathrm{Sinh}[t\,b_j]+t\,b_j)}{b_j^3}, j, k, j, k\right] + \delta\mathrm{aa}\left[\frac{e^{-2\,t\,b_j}\left(-1+e^{t\,b_j}\left(1-t\,b_j\right)\right)}{b_j^2}, i, k, j, k\right];$$

```
Ad[a[t_, j_, k_]][a[1, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
```

$$\mathrm{a}[1, j, l] + \mathrm{ca}[t, l, j, k] + \mathrm{ca}\left[\frac{e^{-t\,b_j}-1}{b_j}, k, j, l\right] + \delta\mathrm{aa}\left[\frac{1-e^{-t\,b_j}-t\,b_j}{b_j^2}, j, k, j, l\right];$$

```
Ad[a[t_, j_, k_]][a[1, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
```

$$\mathrm{a}\left[e^{t\,b_j}, k, l\right] + \mathrm{a}\left[-\frac{\left(-1+e^{t\,b_j}\right)b_k}{b_j}, j, l\right] + \mathrm{ca}\left[\frac{-1+e^{t\,b_j}\left(1-t\,b_j\right)}{b_j}, k, k, l\right] + \mathrm{ca}\left[\frac{b_j-e^{-t\,b_j}\,b_j+b_k+e^{t\,b_j}\left(-1+t\,b_j\right)b_k}{b_j^2}, k, j, l\right] + \mathrm{ca}\left[\frac{b_j+b_k+t\,b_j\,b_k-e^{t\,b_j}\,(b_j+b_k)}{b_j^2}, l, j, k\right] + \delta\mathrm{aa}\left[\frac{1+e^{t\,b_j}\left(-1+t\,b_j\right)}{b_j^2}, j, k, k, l\right] + \delta\mathrm{aa}\left[\frac{1}{b_j^3}e^{-t\,b_j}\left(b_j+e^{2\,t\,b_j}\left(b_j+\left(2-t\,b_j\right)b_k\right)-e^{t\,b_j}\left(2\,b_k+b_j\left(2+t\,b_k\right)\right)\right), j, k, j, l\right];$$

```
Ad[a[t_, j_, k_]][a[1, k_, j_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
```

$$\mathrm{a}\left[e^{t\,b_j}, k, j\right] + \mathrm{a}\left[-1+e^{t\,b_j}, k, k\right] + \mathrm{a}\left[-\frac{\left(-1+e^{t\,b_j}\right)b_k}{b_j}, j, j\right] + \mathrm{a}\left[-\frac{\left(-1+e^{t\,b_j}\right)b_k}{b_j}, j, k\right] + \mathrm{c}\left[\frac{\left(1-e^{t\,b_j}+2\,e^{t\,b_j}\,t\,b_j\right)b_k}{b_j}, j\right] + \mathrm{c}\left[-1+e^{-t\,b_j}+\frac{e^{-t\,b_j}\left(1-e^{2\,t\,b_j}+2\,e^{2\,t\,b_j}\,t\,b_j\right)b_k}{b_j}, k\right] +$$

```
    ca[e^(t b_j) t, k, k, j] + ca[(-1 + e^(t b_j) - 2 e^(t b_j) t b_j)/b_j, j, k, k] + ca[(-1 + e^(t b_j) - e^(t b_j) t b_j)/b_j, k, k, k] +
    ca[(-2 (-1 + e^(t b_j)) b_k + b_j (1 - e^(t b_j) + 2 e^(t b_j) t b_k))/b_j^2, j, j, k] +
    ca[(e^(-t b_j) ((-1 + e^(t b_j)) b_k + b_j (-1 + e^(t b_j) - e^(2 t b_j) t b_k)))/b_j^2, k, j, j] +
    ca[1/b_j^2 e^(-t b_j) (-(1 - 3 e^(t b_j) + 2 e^(2 t b_j)) b_k + b_j (-(-1 + e^(t b_j))^2 + e^(2 t b_j) t b_k)), k, j, k] +
    δa[-e^(t b_j) t, k, j] + δa[(-1 + e^(t b_j) - e^(t b_j) t b_j)/b_j, k, k] + δa[-((1 - e^(t b_j) + e^(t b_j) t b_j) b_k)/b_j^2, j, j] +
    δa[(e^(-t b_j) ((-1 + e^(t b_j)) b_k + b_j (-1 + e^(t b_j) - e^(2 t b_j) t b_k)))/b_j^2, j, k] +
    δaa[(2 - 2 e^(t b_j) + 2 e^(t b_j) t b_j)/b_j^2, j, j, k, k] + δaa[(2 - 2 e^(t b_j) + 2 e^(t b_j) t b_j)/b_j^2, j, k, k, k] +
    δaa[1/b_j^3 e^(-t b_j) ((1 - 4 e^(t b_j) + 3 e^(2 t b_j)) b_k + b_j ((-1 + e^(t b_j))^2 - 2 e^(2 t b_j) t b_k)), j, j, j, k] +
    δaa[1/b_j^3 e^(-t b_j) ((1 - 4 e^(t b_j) + 3 e^(2 t b_j)) b_k + b_j ((-1 + e^(t b_j))^2 - 2 e^(2 t b_j) t b_k)), j, k, j, k];
Ad[a[t_, j_, k_]][c[1, i_]] /; FreeQ[t, b_] ∧ ({j, k} ⋂ {i} === {}) := c[1, i];
Ad[a[t_, j_, k_]][c[1, j_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[1, j] + c[1 - e^(-t b_j), k] + δa[(e^(-t b_j) - 1)/b_j, j, k];
Ad[a[t_, j_, k_]][c[1, k_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[e^(-t b_j), k] + δa[(1 - e^(-t b_j))/b_j, j, k];

Ad[x_β | x_c | x_δa | x_ca | x_δaa][y_] := y + B[x, y];

Ad[x_][a[f_, i_, j_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][a[1, i, j]];
Ad[x_][c[f_, i_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][c[1, i]];
Ad[x_][δa[f_, j_, k_]] := δ ** (β[f] ** Ad[x][a[1, j, k]]);
Ad[x_][ca[f_, i_, j_, k_]] := Ad[x][c[f, i]] ** Ad[x][a[1, j, k]];
Ad[x_][δaa[f_, i_, j_, k_, l_]] := Ad[x][δa[f, i, j]] ** Ad[x][a[1, k, l]];
Ad[x_][y_Plus] := Ad[x] /@ y;

Ad::ndef = "Ad[`1` is not defined on `2`.";
Ad[x_][y_] := (Message[Ad::ndef, x, y]; Undefined);
```

AutoAd - Ad tests

```
Module[{t1, t2},
   {t1 = S[AutoAd[a[t, j, k]][#]],
    S[Ad[a[t, j, k]][#] - t1]}
  ] & @ a[1, i, k]
```

$$\left\{a\left[e^{-t\,b_j},\,i,\,k\right]+a\left[\frac{\left(1-e^{-t\,b_j}\right)b_i}{b_j},\,j,\,k\right]+\right.$$
$$ca\left[\frac{e^{-2\,t\,b_j}\,b_i\left(-1+e^{2\,t\,b_j}-2\,e^{t\,b_j}\,t\,b_j\right)}{b_j^2},\,k,\,j,\,k\right]+ca\left[\frac{e^{-2\,t\,b_j}\left(1-e^{t\,b_j}+e^{t\,b_j}\,t\,b_j\right)}{b_j},\,k,\,i,\,k\right]+$$
$$\delta a\left[-\frac{e^{-2\,t\,b_j}\,b_i\left(1-e^{t\,b_j}+e^{t\,b_j}\,t\,b_j\right)}{b_j^2},\,j,\,k\right]+\delta a\left[\frac{e^{-2\,t\,b_j}\left(1-e^{t\,b_j}+e^{t\,b_j}\,t\,b_j\right)}{b_j},\,i,\,k\right]+$$
$$\delta aa\left[-\frac{e^{-2\,t\,b_j}\,b_i\left(-1+e^{2\,t\,b_j}-2\,e^{t\,b_j}\,t\,b_j\right)}{b_j^3},\,j,\,k,\,j,\,k\right]+$$
$$\left.\delta aa\left[\frac{e^{-2\,t\,b_j}\left(-1+e^{t\,b_j}-e^{t\,b_j}\,t\,b_j\right)}{b_j^2},\,i,\,k,\,j,\,k\right],\,0\right\}$$

```
AdTests[a[t, j, k]] =
  {β[f[b_j, b_k]], a[1, j, k], a[1, n, i], a[1, j, j], a[1, k, k], c[1, i],
   c[1, j], c[1, k], a[1, j, l], a[1, i, j], a[1, i, k], a[1, k, l], a[1, k, j]};
```

```
S[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ Take[AdTests[a[t, j, k]], All]
```

```
$Aborted
```

The semi group properties

```
Module[{t1, t2, t3, t4},
    t1 = Ad[a[t, j, k]][#] /.
      (h : (β | a | c | δa | ca | δaa))[c_, r___] :→ h[SeriesCoefficient[c, {t, 0, 1}], r];
    t2 = B[a[1, j, k], #];
    t3 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
    t4 = # // Ad[a[t + s, j, k]];
    # → S[{t1 == t2, t3 - t4}]
  ] & /@ AdTests[a[t, j, k]] // ColumnForm
```

```
β[f[b_j, b_k]] → {True, 0}
a[1, j, k] → {True, 0}
a[1, n, i] → {True, 0}
a[1, j, j] → {True, 0}
a[1, k, k] → {True, 0}
c[1, i] → {True, 0}
c[1, j] → {True, 0}
c[1, k] → {True, 0}
a[1, j, l] → {True, 0}
a[1, i, j] → {True, 0}
a[1, i, k] → {True, 0}
a[1, k, l] → {True, 0}
a[1, k, j] → {True, 0}
```

R

```
Switch[10,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t bi, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[bi, bj]]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bj] + bi f1[bj]]],
  4, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[c[f3[bi, bj], i]] // Ad[c[f4[bi, bj], j]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f5[bi], i, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f6[bi, bj], j, i, j]],
  7, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[ca[f5[bi], i, i, j]] // Ad[ca[f6[bi, bj], j, i, j]],
  8, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[δa[f2[bi, bj], j, i]],
  9, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[ca[f7[bi, bj], i, j, i]] // Ad[ca[f8[bi, bj], j, j, i]],
  10, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f8[bi, bj], j, j, i]]
 ];
VerifyR3[expr_] := Module[{lhs, rhs},
  lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // S;
  rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // S;
  expr → S[lhs - rhs] == 0
 ]
```

Verifying R3

```
Print /@
  VerifyR3 /@ {β[f[b1, b2, b3, b4]], c[f[b1, b2, b3, b4], 1], c[f[b1, b2, b3, b4], 2],
    c[f[b1, b2, b3, b4], 3], δa[f[b1, b2, b3, b4], 1, 2], δa[f[b1, b2, b3, b4], 1, 3],
    δa[f[b1, b2, b3, b4], 2, 3], a[f[b1, b2, b3, b4], 1, 4], a[f, 2, 4]};
```

$\beta[f[b_1, b_2, b_3, b_4]] \to \text{True}$

$c[f[b_1, b_2, b_3, b_4], 1] \to \text{True}$

$c[f[b_1, b_2, b_3, b_4], 2] \to \text{True}$

$c[f[b_1, b_2, b_3, b_4], 3] \to \text{True}$

$\delta a[f[b_1, b_2, b_3, b_4], 1, 2] \to \text{True}$

$\delta a[f[b_1, b_2, b_3, b_4], 1, 3] \to \text{True}$

$\delta a[f[b_1, b_2, b_3, b_4], 2, 3] \to \text{True}$

$$a[f[b_1, b_2, b_3, b_4], 1, 4] \to$$
$$ca[e^{-b_2}(-1+e^{b_2})\, f[b_1, b_2, b_3, b_4]\,(f8[b_1, b_2]\, b_2 - f8[b_1, b_3]\, b_3), 4, 1, 3] +$$
$$ca\left[-\frac{1}{b_2} e^{-b_2}(-1+e^{b_2})\, f[b_1, b_2, b_3, b_4]\, b_1\,(f8[b_1, b_2]\, b_2 - f8[b_1, b_3]\, b_3), 4, 2, 3\right] +$$
$$\delta aa\left[-\frac{(-1+e^{b_2})\, f[b_1, b_2, b_3, b_4]\, f8[b_1, b_3]\, b_1}{b_2}, 2, 3, 3, 4\right] +$$
$$\delta aa\left[-\frac{1}{b_2} e^{-b_2}(-1+e^{b_2})\, f[b_1, b_2, b_3, b_4]\,(f8[b_1, b_2]\, b_2 - f8[b_1, b_3]\, b_3), 1, 3, 2, 4\right] + \delta aa\left[\frac{1}{b_2^2}\right.$$
$$\left. e^{-b_2}(-1+e^{b_2})\, f[b_1, b_2, b_3, b_4]\, b_1\,(f8[b_1, b_2]\, b_2 + (-1+e^{b_2})\, f8[b_1, b_3]\, b_3), 2, 3, 2, 4\right] == 0$$

$$a[f, 2, 4] \to ca[(-1+e^{b_1})\, f\,(f8[b_1, b_3] - f8[b_2, b_3])\, b_2, 3, 3, 4] +$$
$$ca[e^{-b_2}(-1+e^{b_1})\, f\,((-2+e^{b_2})\, f8[b_1, b_2]\, b_2 - (-1+e^{b_2})\, f8[b_1, b_3]\, b_3), 4, 2, 3] +$$
$$ca\left[-\frac{1}{b_1} e^{-b_2}(-1+e^{b_1})\, f\, b_2\,((-2+e^{b_2})\, f8[b_1, b_2]\, b_2 - (-1+e^{b_2})\, f8[b_1, b_3]\, b_3), 4, 1, 3\right] +$$
$$ca\left[-\frac{1}{b_1^2} e^{-b_2}\, f\, b_2\,(2(-1+e^{b_1})(-1+e^{b_2}) + b_1(-(1+e^{b_1})(-1+e^{b_2}) + e^{b_2}(-1+e^{b_1})\, f8[b_1, b_3]\, b_3)),\right.$$
$$\left. 3, 1, 4\right] + ca\left[\frac{1}{b_1}\right.$$
$$\left. e^{-b_2}\, f\,(2(-1+e^{b_1})(-1+e^{b_2}) + b_1(-(1+e^{b_1})(-1+e^{b_2}) + e^{b_2}(-1+e^{b_1})\, f8[b_2, b_3]\, b_3)), 3, 2, 4\right] +$$
$$\delta aa[(-1+e^{b_1})(-1+e^{b_2})\, f\, f8[b_1, b_3], 2, 3, 3, 4] + \delta aa\left[\frac{(-1+e^{b_1})\, f\, f8[b_2, b_3]\, b_2}{b_1}, 1, 3, 3, 4\right] +$$
$$\delta aa\left[-\frac{e^{-b_2}(-1+e^{b_2})\, f\,(2 - 2e^{b_1} + (1+e^{b_1})\, b_1)\, b_2}{b_1^3}, 1, 3, 1, 4\right] +$$
$$\delta aa\left[-\frac{1}{b_2} e^{-b_2}(-1+e^{b_1})\, f\left((-2+e^{b_2})\, f8[b_1, b_2]\, b_2 + (-1+e^{b_2})^2\, f8[b_1, b_3]\, b_3\right), 2, 3, 2, 4\right] +$$
$$\delta aa\left[\frac{1}{b_1^2} e^{-b_2}\, f\,(-2(-1+e^{b_1})(-1+e^{b_2}) +\right.$$
$$b_1(-1 - e^{b_1} + e^{b_2} + e^{b_1+b_2} + (-1+e^{b_1})(-2+e^{b_2})\, f8[b_1, b_2]\, b_2 - (-1+e^{b_1})(-1+e^{b_2})$$
$$\left. f8[b_1, b_3]\, b_3 + e^{b_2}\, f8[b_2, b_3]\, b_3 - e^{b_1+b_2}\, f8[b_2, b_3]\, b_3)), 1, 3, 2, 4\right] == 0$$

```
Total[MapIndexed[(#1 /. f → f#2[[1]]) &, DeleteCases[FormalBasis[{j, k}, f], _β | _a]]]
```

```
c[f1[bj, bk], j] + c[f2[bj, bk], k] + ca[f7[bj, bk], j, j, j] + ca[f8[bj, bk], j, j, k] +
 ca[f9[bj, bk], j, k, j] + ca[f10[bj, bk], j, k, k] + ca[f11[bj, bk], k, j, j] +
 ca[f12[bj, bk], k, j, k] + ca[f13[bj, bk], k, k, j] + ca[f14[bj, bk], k, k, k] +
 δa[f3[bj, bk], j, j] + δa[f4[bj, bk], j, k] + δa[f5[bj, bk], k, j] + δa[f6[bj, bk], k, k] +
 δaa[f15[bj, bk], j, j, j, j] + δaa[f16[bj, bk], j, j, j, k] + δaa[f17[bj, bk], j, j, k, j] +
 δaa[f18[bj, bk], j, j, k, k] + δaa[f19[bj, bk], j, k, j, k] + δaa[f20[bj, bk], j, k, k, k] +
 δaa[f21[bj, bk], k, j, k, j] + δaa[f22[bj, bk], k, j, k, k] + δaa[f23[bj, bk], k, k, k, k]
```

```
f22[__] = 0; f21[__] = 0; f9[__] = 0; f5[__] = 0; f13[__] = 0; f17[__] = 0; f7[__] = 0;
f8[__] = 0;
f10[bj_, bk_] := g2[bk];
f1[bj_, bk_] := g3[bk];
f12[bj_, bk_] := -bj f19[bj, bk];
f19[bj_, bk_] := -(e^(-bj) (2 - 2 e^bj + bj + e^bj bj))/(2 bj^3);
f16[bj_, bk_] := g1[bj];
f11[bj_, bk_] := g4[bj];
f14[bj_, bk_] := -bj f20[bj, bk];
f20[bj_, bk_] := (-1 + e^bj)/(e^bj bj) (-2 + 2 e^bk - bk - e^bk bk + 2 bk^2 g2[bk] + 2 e^bk bk^2 g4[bk])/(2 (-1 + e^bk) bk^2);
f2[bj_, bk_] := g5[bj] - bj f4[bj, bk];
(* Non-forced choice: *) f4[__] = 0;
g5[bj_] := -(e^(-bj) (2 - 2 e^bj + bj + e^bj bj + 2 bj g3[bj]))/(2 bj);
(* Non-forced choices: *) g1[_] = 0; g2[_] = 0; g3[_] = 0;
g4[_] = 0; f3[__] = 0; f6[__] = 0; f15[__] = 0; f18[__] = 0; f23[__] = 0;
Total[
  MapIndexed[(#1 /. f → f#2[[1]]) &, DeleteCases[FormalBasis[{j, k}, f], _β | _a]]] // S
```

$$c\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j},k\right]+ca\left[\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^2},k,j,k\right]+$$
$$ca\left[\frac{e^{-b_j}\left(-1+e^{b_j}\right)\left(2-2e^{b_k}+\left(1+e^{b_k}\right)b_k\right)}{2\left(-1+e^{b_k}\right)b_k^2},k,k,k\right]+$$
$$\delta aa\left[-\frac{e^{-b_j}\left(2-2e^{b_j}+\left(1+e^{b_j}\right)b_j\right)}{2b_j^3},j,k,j,k\right]+$$
$$\delta aa\left[-\frac{e^{-b_j}\left(-1+e^{b_j}\right)\left(2-2e^{b_k}+\left(1+e^{b_k}\right)b_k\right)}{2\left(-1+e^{b_k}\right)b_j\,b_k^2},j,k,k,k\right]$$

```
R[j_, k_][x_] :=
 x // Ad[a[1, j, k]] // (# + B[Total[MapIndexed[(#1 /. f → f#2[[1]]) &, DeleteCases[
        FormalBasis[{j, k}, f], _β | _a]]], #]) &
```

```
Print /@ VerifyR3 /@
  {a[f[b1, b2, b3, b4], 1, 4], a[f[b1, b2, b3, b4], 2, 4], a[f[b1, b2, b3, b4], 3, 4],
   a[f[b1, b2, b3, b4], 4, 1], a[f[b1, b2, b3, b4], 4, 2], a[f[b1, b2, b3, b4], 4, 3]};
```

$a[f[b_1, b_2, b_3, b_4], 1, 4] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 2, 4] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 3, 4] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 4, 1] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 4, 2] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 4, 3] \to \text{True}$

```
Print[VerifyR3[#]] & /@
  {a[f[b1, b2, b3, b4], 1, 2], a[f[b1, b2, b3, b4], 1, 3], a[f[b1, b2, b3, b4], 2, 3],
   a[f[b1, b2, b3, b4], 2, 1], a[f[b1, b2, b3, b4], 3, 1], a[f[b1, b2, b3, b4], 3, 2]};
```

$a[f[b_1, b_2, b_3, b_4], 1, 2] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 1, 3] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 2, 3] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 2, 1] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 3, 1] \to \text{True}$

$a[f[b_1, b_2, b_3, b_4], 3, 2] \to \text{True}$