Pensieve header: One-Co computations in the abc presentation; continues pensieve://2015-07/.

```
{ϵ1 = -1, ϵ2 = -1, ϵ6 = -1, ϵ7 = 1};
```

The bracket

On the elements β, a, c, δa, ca, δaa.

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];

Simp[expr_] := Simplify[expr];
S[β[f_]] := β[Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[δa[f_, i_, j_]] := δa[Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[δaa[f_, i_, j_, k_, l_]] := δaa[Simp[f], i, j, k, l];
S[expr_] := expr /. (λ_β | λ_a | λ_δa | λ_c | λ_ca | λ_δaa) :→ S[λ];

β[0] := 0;
β /: β[f_] + β[g_] := β[f + g] // S;
β /: g_ * β[f_] := β[g f] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f + g, j, k] // S;
a /: g_ * a[f_, j_, k_] := a[g f, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f + g, j] // S;
c /: g_ * c[f_, j_] := c[g f, j] // S;
δa[0, _, _] := 0;
δa /: δa[f_, j_, k_] + δa[g_, j_, k_] := δa[f + g, j, k] // S;
δa /: g_ * δa[f_, j_, k_] := δa[g f, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f + g, j, k, l] // S;
ca /: g_ * ca[f_, j_, k_, l_] := ca[g f, j, k, l] // S;
δaa[0, _, _, _, _] := 0;
δaa /: δaa[f_, i_, j_, k_, l_] + δaa[g_, i_, j_, k_, l_] :=
  δaa[f + g, i, j, k, l] // S;
δaa /: g_ * δaa[f_, i_, j_, k_, l_] := δaa[g f, i, j, k, l] // S;
```

δaa relations

```
VS = False;
```

"First sort tails then sort heads"

Standard Swinging - sorts heads, if support is 4 strands:

```
S[δaa[f_, i_, j_, k_, l_]] /; DQ[i, j, k, l] ⋀ OQ[i, k] ⋀ ! OQ[j, l] := (
   If[VS, Print["Standard swinging on ", δaa[f, i, j, k, l]]];
   S[δaa[f, i, l, k, j] + ca[b_k f, l, i, j] -
     ca[b_i f, l, k, j] - ca[b_k f, j, i, l] + ca[b_i f, j, k, l]]
  );
```

Locality - sorts tails when supports are disjoint:

```
S[δaa[f_, i_, j_, k_, l_]] /; ({i, j} ⋂ {k, l} === {}) ⋀ ! OQ[i, k] := (
   If[VS, Print["Locality on ", δaa[f, i, j, k, l]]];
   δaa[f, k, l, i, j] // S
  );
```

Commute Heads - sorts tails when the heads are the same:

```
S[δaa[f_, i_, k_, j_, k_]] /; DQ[i, j, k] ⋀ ! OQ[i, j] := (
   If[VS, Print["Commute heads on ", δaa[f, i, k, j, k]]];
   S[Expand[
     δaa[f, j, k, i, k] + ϵ2 (δa[b_i f, j, k] - δa[b_j f, i, k])
    ]]
  );
```

Commute Head/Tail - sorts tails:

```
S[δaa[f_, i_, j_, k_, i_]] /; DQ[i, j, k] ⋀ ! OQ[i, k] := (
   If[VS, Print["Commute head/tail on ", δaa[f, i, j, k, i]]];
   S[
    δaa[f, k, i, i, j] + δaa[f, k, j, i, j] - δaa[f, i, j, k, j]
   ]
  );
```

Commute Head/Tail - sorts heads where heads & tails are both broken:

```
S[δaa[f_, k_, j_, j_, i_]] /; DQ[i, j, k] ⋀ OQ[i, j, k] := (
   If[VS, Print["Commute head/tail on ", δaa[f, k, j, j, i]]];
   S[
    δaa[f, j, i, k, j] + δaa[f, j, i, k, i] - δaa[f, k, i, j, i]
   ]
  );
```

2113 Swinging - sorts tails:

```
δaa[f, j, i, ii, k] // S
```

```
Locality on δaa[f, j, i, ii, k]
Standard swinging on δaa[f, ii, k, j, i]
ca[-f b_ii, i, j, k] + ca[f b_ii, k, j, i] +
 ca[-f b_j, k, ii, i] + ca[f b_j, i, ii, k] + δaa[f, ii, i, j, k]
```

```
S[δaa[f_, j_, i_, i_, k_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["2113 swinging on ", δaa[f, j, i, i, k]]];
   S[ca[-f b_i, i, j, k] + ca[f b_i, k, j, i] +
     c[-f b_j, k] ** aop[1, i] + ca[f b_j, i, i, k] + aop[f, i] ** δa[1, j, k]]
  );
```

```
δaa[f, j, i, i, k] // S
```

```
c[-f b_i b_j ∈_1, k] + ca[-f b_i, i, j, k] + ca[f b_i, k, j, i] +
 ca[-f b_j, k, i, i] + ca[f b_j, i, i, k] + δa[f b_i ∈_1, j, k] + δaa[f, i, i, j, k]
```

3112 Swinging - sorts tails:

```
δaa[f, k, i, ii, j] // S
```

```
Locality on δaa[f, k, i, ii, j]
Standard swinging on δaa[f, ii, j, k, i]
ca[-f b_ii, i, k, j] + ca[f b_ii, j, k, i] +
 ca[-f b_k, j, ii, i] + ca[f b_k, i, ii, j] + δaa[f, ii, i, k, j]
```

```
S[δaa[f_, k_, i_, i_, j_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["3112 swinging on ", δaa[f, k, i, i, j]]];
   S[ca[-f b_i, i, k, j] + ca[f b_i, j, k, i] +
     c[-f b_k, j] ** aop[1, i] + ca[f b_k, i, i, j] + aop[1, i] ** δa[f, k, j]]
  );
```

```
δaa[f, k, i, i, j] // S
```

```
3112 swinging on δaa[f, k, i, i, j]
c[-f b_i b_k ∈_1, j] + ca[-f b_i, i, k, j] + ca[f b_i, j, k, i] +
 ca[-f b_k, j, i, i] + ca[f b_k, i, i, j] + δa[f b_i ∈_1, k, j] + δaa[f, i, i, k, j]
```

Tails Commute - sorts heads when the tails are the same:

```
S[δaa[f_, i_, j_, i_, l_]] /; DQ[i, j, l] ∧ ! OQ[j, l] := (
   If[VS, Print["Tails commute on ", δaa[f, i, j, i, l]]];
   δaa[f, i, l, i, j] // S
  );
```

1321 Swinging - sorts heads:

```
δaa[f, i, k, j, ii] // S
```

```
Standard swinging on δaa[f, i, k, j, ii]
```

$ca[-f\, b_i, ii, j, k] + ca[f\, b_i, k, j, ii] + ca[-f\, b_j, k, i, ii] + ca[f\, b_j, ii, i, k] + \delta aa[f, i, ii, j, k]$

```
S[δaa[f_, i_, k_, j_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1321 swinging on ", δaa[f, i, k, j, i]]];
   S[ca[-f b_i, i, j, k] + ca[f b_i, k, j, i] +
     ca[-f b_j, k, i, i] + ac[f b_j, i, k, i] + δaa[f, i, i, j, k]]
  );
```

1322 Swinging - sorts heads, but breaks tails:

```
δaa[f, i, k, j, jj] // S
```

```
Standard swinging on δaa[f, i, k, j, jj]
```

$ca[-f\, b_i, jj, j, k] + ca[f\, b_i, k, j, jj] + ca[-f\, b_j, k, i, jj] + ca[f\, b_j, jj, i, k] + \delta aa[f, i, jj, j, k]$

```
S[δaa[f_, i_, k_, j_, j_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1322 swinging on ", δaa[f, i, k, j, j]]];
   S[ac[-f b_i, j, k, j] + ca[f b_i, k, j, j] +
     ca[-f b_j, k, i, j] + ca[f b_j, j, i, k] + δaa[f, j, k, i, j]]
  );
```

1332 Swinging - sorts heads:

```
δaa[f, i, k, kk, j] // S
```

```
Standard swinging on δaa[f, i, k, kk, j]
```

$ca[-f\, b_i, j, kk, k] + ca[f\, b_i, k, kk, j] + ca[-f\, b_{kk}, k, i, j] + ca[f\, b_{kk}, j, i, k] + \delta aa[f, i, j, kk, k]$

```
S[δaa[f_, i_, k_, k_, j_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1332 swinging on ", δaa[f, i, k, k, j]]];
   S[c[-f b_i, j] ** aop[1, k] + ca[f b_i, k, k, j] +
     ca[-f b_k, k, i, j] + ca[f b_k, j, i, k] + δa[f, i, j] ** aop[1, k]]
  );
```

```
δaa[f, i, k, k, j] // S
```

```
1332 swinging on δaa[f, i, k, k, j]
```

$ca[-f\, b_i, j, k, k] + ca[f\, b_i, k, k, j] + ca[-f\, b_k, k, i, j] + ca[f\, b_k, j, i, k] + \delta a[f\, b_k\, \epsilon_1, i, j] + \delta aa[f, i, j, k, k]$

1231 Swinging - sorts heads:

```
δaa[f, i, j, k, ii] // S
```

```
Standard swinging on δaa[f, i, j, k, ii]
```

```
ca[-f b_i, ii, k, j] + ca[f b_i, j, k, ii] +
 ca[-f b_k, j, i, ii] + ca[f b_k, ii, i, j] + δaa[f, i, ii, k, j]
```

```
S[δaa[f_, i_, j_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["1231 swinging on ", δaa[f, i, j, k, i]]];
   S[ac[-f b_i, k, j, i] + ca[f b_i, j, k, i] +
     ca[-f b_k, j, i, i] + ac[f b_k, i, j, i] + δaa[f, i, i, k, j]]
  );
```

1211 sliding - sorts heads:

```
S[δaa[f_, i_, j_, i_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["1211 sliding on ", δaa[f, i, j, i, i]]];
   S[δaa[f, i, i, i, j]]
  );
```

2111 sliding - sorts tails:

```
S[δaa[f_, j_, i_, i_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2111 sliding on ", δaa[f, j, i, i, i]]];
   S[δaa[f, i, i, j, i]]
  );
```

2212 sliding - sorts tails:

```
S[δaa[f_, j_, j_, i_, j_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2212 sliding on ", δaa[f, j, j, i, j]]];
   S[δaa[f, i, j, j, j]]
  );
```

2221 sliding - sorts heads:

```
S[δaa[f_, j_, j_, j_, i_]] /; DQ[i, j] ∧ OQ[i, j] := (
   If[VS, Print["2221 sliding on ", δaa[f, j, j, j, i]]];
   S[δaa[f, j, i, j, j]]
  );
```

2231 Swinging - sorts heads:

```
δaa[f, j, jj, k, i] // S
```

```
Standard swinging on δaa[f, j, jj, k, i]
```

```
ca[-f b_j, i, k, jj] + ca[f b_j, jj, k, i] +
 ca[-f b_k, jj, j, i] + ca[f b_k, i, j, jj] + δaa[f, j, i, k, jj]
```

```
S[δaa[f_, j_, j_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["2231 swinging on ", δaa[f, j, j, k, i]]];
   S[ca[-f b_j, i, k, j] + ca[f b_j, j, k, i] +
     ac[-f b_k, j, i, j] + ca[f b_k, i, j, j] + δaa[f, j, i, k, j]]
  );
```

2331 Swinging - sorts heads:

```
δaa[f, j, k, kk, i] // S
```

```
ca[-f b_j, i, kk, k] + ca[f b_j, k, kk, i] +
 ca[-f b_kk, k, j, i] + ca[f b_kk, i, j, k] + δaa[f, j, i, kk, k]
```

```
S[δaa[f_, j_, k_, k_, i_]] /; DQ[i, j, k] ∧ OQ[i, j, k] := (
   If[VS, Print["2331 swinging on ", δaa[f, j, k, k, i]]];
   S[c[-f b_j, i] ** aop[1, k] + ca[f b_j, k, k, i] +
     ca[-f b_k, k, j, i] + ca[f b_k, i, j, k] + δa[f, j, i] ** aop[1, k]]
  );
```

Backie jkkj Swinging - sorts tails or heads:

```
δaa[f, j, k, kk, jj] // S
```

```
ca[-f b_j, jj, kk, k] + ca[f b_j, k, kk, jj] +
 ca[-f b_kk, k, j, jj] + ca[f b_kk, jj, j, k] + δaa[f, j, jj, kk, k]
```

```
S[δaa[f_, j_, k_, k_, j_]] /; DQ[j, k] := (
   If[VS, Print["Backie swinging on ", δaa[f, j, k, k, j]]];
   S[c[-f b_j, j] ** aop[1, k] + ca[f b_j, k, k, j] +
     ca[-f b_k, k, j, j] + ac[f b_k, j, k, j] + δa[f, j, j] ** aop[1, k]]
  );
```

NonCommutativeMultiply

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_, 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;
```

```
β[f_] ** a[g_, j_, k_] := a[f g, j, k];
β[f_] ** c[g_, j_] := c[f g, j];
c[g_, j_] ** β[f_] := c[f g, j];
β[f_] ** δa[g_, j_, k_] := δa[f g, j, k];
β[f_] ** ca[g_, i_, j_, k_] := ca[f g, i, j, k];
ca[g_, i_, j_, k_] ** β[f_] := ca[f g, i, j, k];
β[f_] ** δaa[g_, i_, j_, k_, l_] := δaa[f g, i, j, k, l];
δa[g_, j_, k_] ** β[f_] := δa[f g, j, k];
δ ** a[f_, i_, j_] := δa[f, i, j];
c[f_, i_] ** a[g_, j_, k_] := ca[f g, i, j, k];
a[f_, i_, j_] ** δa[g_, k_, l_] := δaa[f g, i, j, k, l];
δa[f_, i_, j_] ** a[g_, k_, l_] := δaa[f g, i, j, k, l];
```

```
δ ** _c = 0;
δ ** _δa = 0;
δ ** _ca = 0;
δ ** _δaa = 0;
_c ** _c = 0;
_c ** _δa = _δa ** _c = 0;
_c ** _ca = _ca ** _c = 0;
_c ** _δaa = _δaa ** _c = 0;
_δa ** _δa = 0;
_δa ** _δaa = _δaa ** _δa = 0;
_δa ** _ca = _ca ** _δa = 0;
```

```
NonCommutativeMultiply::ndef =
  "NonCommuatativeMultiply is not defined on {`1`,`2`}."
NonCommutativeMultiply[x_, y_] :=
  (Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
```

```
NonCommuatativeMultiply is not defined on {`1`,`2`}.
```

Bracket Generalities

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

The γ shortcuts

```
γ[f_, j_, k_] := δa[f, j, k] - c[b_j f, k] // S;
γ[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;
ac[f_, j_, k_, l_] := ca[f, l, j, k] + B[a[1, j, k], c[f, l]];
aop[f_, j_] := a[f, j, j] + β[ϵ1 f b_j] + c[ϵ1 f, j];
```

Fundamental Brackets

a-β, a-c, a-a, AS

```
B[a[j_, k_], β[g_]] := γ[∂_{b_j} g - ∂_{b_k} g, j, k];
B[β[g_], a[j_, k_]] := -B[a[j, k], β[g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k} ⋂ {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // S;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] - γ[1, j, k, l] // S;
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
(* backie *) B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[b_j, k, j] - a[b_k, j, k] + a[b_j, k, k] - a[b_k, j, j] + ca[1, k, k, j] -
   ca[1, j, j, k] + ca[1, k, j, j] - ca[1, j, k, k] + γ[1, j, k] - γ[1, k, j];
(* [tail, selfie] *) B[a[j_, k_], a[j_, j_]] /; DQ[j, k] := γ[-ϵ6, j, k] // S;
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
(* [head, selfie] *) B[a[j_, k_], a[k_, k_]] /; DQ[j, k] := γ[-ϵ7, j, k] // S;
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[k, k]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] := γ[-f g, j, k];
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] := γ[f g, j, k];
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k} ⋂ {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];
```

Vanishing brackets

```
B[_β, _β | δ | _c | _δa | _ca | _δaa] = 0;
B[_β | δ | _c | _δa | _ca | _δaa, _β] = 0;
B[δ | _c | _δa | _ca | _δaa, δ | _c | _δa | _ca | _δaa] = 0;
```

Composite Brackets

```
B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[1, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[1, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[1, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[1, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];
```

```
B::ndef = "B is not defined on {`1`,`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
```

```
B is not defined on {`1`,`2`}.
```

Testing Jacobi and Anti-Symmetry

```
FormalBasis[n_, f_] := Module[{ff},
   ff = f @@ Table[b_i, {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, i, j], {i, n}, {j, n}],
      Table[c[ff, i], {i, n}],
      Table[δa[ff, i, j], {i, n}, {j, n}],
      Table[ca[ff, i, j, k], {i, n}, {j, n}, {k, n}],
      Table[δaa[ff, i, j, k, l], {i, n}, {j, n}, {k, i, n}, {l, j, n}]
     } /. 1[___] → 1
  ];
FormalPlusBasis[n_, f_] := Module[{ff},
   ff = f @@ Table[b_i, {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[c[ff, i], {i, n}],
      Table[δa[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[ca[ff, i, j, k], {i, n}, {j, n - 1}, {k, j + 1, n}],
      Table[δaa[ff, i, j, k, l], {i, n - 1}, {j, i + 1, n}, {k, n - 1}, {l, k + 1, n}]
     } /. 1[___] → 1
  ];

VS = False;
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} →  as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalPlusBasis[3, f],
   FormalPlusBasis[3, g]
  ]], 0]
```

```
{}
```

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalBasis[3, f],
   FormalBasis[3, g]
  ]], 0]
```

{}

```
Jacobi[x1_, x2_, x3_] := Module[{Jac},
   Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
   If[Jac === 0, Jac, {x1, x2, x3} → Jac]
  ];
```

```
JacPlusErrors = DeleteCases[
  bas1 = FormalPlusBasis[4, f];
  bas2 = FormalPlusBasis[4, g];
  bas3 = FormalPlusBasis[4, h];
  Flatten[
   Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]]],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

{}

```
JacPlusErrors // Length
```

0

```
VS = False;
JacErrors = DeleteCases[
  bas1 = FormalBasis[4, f];
  bas2 = FormalBasis[4, g];
  bas3 = FormalBasis[4, h];
  Flatten[
   Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]]],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

{}

The Adjoint action

AutoAd

```
AutoAd[x_][y_] :=
  Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
   pows = NestList[B[x, #] &, y, 20];
   Print["pows computed for ", {x, y}, "..."];
   states = Union[
     Cases[pows, s_β | s_a | s_c | s_δa | s_ca | s_δaa :→ ReplacePart[s, 1 → _], ∞]];
   Sum[
    seq = Cases[{#}, states⟦i⟧, ∞] & /@ pows;
    seq = Replace[seq, {{_[f_, ___]} :→ f, {} → 0}, {1}];
    Print["seq computed... ", states⟦i⟧, " is ", i, "/", Length@states];
    dseq = Drop[seq, sh];
    If[Union[Length[MonomialList[#]] & /@ dseq] === {1} ∧
      Union[Length[FactorTermsList[#]] & /@ dseq] === {2},
     sf1 = FindSequenceFunction[FactorTermsList[#]⟦1⟧ & /@ dseq];
     sf2 = FindSequenceFunction[FactorTermsList[#]⟦2⟧ & /@ dseq];
     Print["sf1: ", sf1, " sf2: ", sf2];
     sf = (sf1[#] sf2[#] &),
     (*Else*) sf = FindSequenceFunction[dseq,
       FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
         "Polynomial", "RationalFunction", "HypergeometricTerm"}];
     Print["sf: ", sf];
    ];
    ReplacePart[states⟦i⟧, 1 → Simplify[Sum[seq⟦n + 1⟧/n!, {n, 0, sh-1}] + Sum[sf[n + 1 - sh]/n!, {n, sh, ∞}]]],
    {i, Length@states}]
  ];
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)

AutoAd[a[t, j, k]][a[1, k, k]]
```

The penultimate expression, as displayed:

$$\text{ReplacePart}\left[\text{states}⟦i⟧,\ 1 \to \text{Simplify}\left[\sum_{n=0}^{sh-1} \frac{\text{seq}⟦n+1⟧}{n!} + \sum_{n=sh}^{\infty} \frac{\text{sf}[n+1-sh]}{n!}\right]\right],$$

```
pows computed for {a[t, j, k], a[1, k, k]}...
seq computed... a[_, k, k] is 1/3
sf1: 1 & sf2: 0 &
seq computed... c[_, k] is 2/3
```

sf1: $(-1)^{1+\#1}$ & sf2: $t^4 b_j^4 (t b_j)^{\#1}$ &

```
seq computed... δa[_, j, k] is 3/3
```

sf1: $(-1)^{\#1}$ & sf2: $t^4 b_j^3 (t b_j)^{\#1}$ &

$$a[1, k, k] + c\left[1 - e^{-t b_j}, k\right] + \delta a\left[\frac{-1 + e^{-t b_j}}{b_j}, j, k\right]$$

Ad

$$\text{Ad}[a[t\_, j\_, k\_]][\beta[f\_]] \;/;\; \text{FreeQ}[t, b\_] :=$$
$$\beta[f] + c\left[\left(1 - e^{-t b_j}\right)(\partial_{b_k} f - \partial_{b_j} f), k\right] + \delta a\left[\frac{\left(e^{-t b_j} - 1\right)(\partial_{b_k} f - \partial_{b_j} f)}{b_j}, j, k\right];$$

$$\text{Ad}[a[t\_, j\_, k\_]][a[1, j\_, k\_]] \;/;\; \text{FreeQ}[t, b\_] := a[1, j, k];$$

$$\text{Ad}[a[t\_, j\_, k\_]][a[1, n\_, i\_]] \;/;$$
$$\text{FreeQ}[t, b\_] \bigwedge \left(\{j, k\} \bigcap \{n, i\} === \{\}\right) := a[1, n, i];$$

$$\text{Ad}[a[t\_, j\_, k\_]][a[1, j\_, j\_]] \;/;\; \text{DQ}[j, k] \bigwedge \text{FreeQ}[t, b\_] :=$$
$$a[1, j, j] + c\left[-1 + e^{-t b_j}, k\right] + \delta a\left[\frac{1 - e^{-t b_j}}{b_j}, j, k\right];$$

$$\text{Ad}[a[t\_, j\_, k\_]][a[1, k\_, k\_]] \;/;\; \text{DQ}[j, k] \bigwedge \text{FreeQ}[t, b\_] :=$$
$$a[1, k, k] + c\left[1 - e^{-t b_j}, k\right] + \delta a\left[\frac{-1 + e^{-t b_j}}{b_j}, j, k\right];$$

$$\text{Ad}[a[t\_, j\_, k\_]][a[1, i\_, j\_]] \;/;\; \text{DQ}[i, j, k] \bigwedge \text{FreeQ}[t, b\_] :=$$
$$a[1, i, j] + a\left[1 - e^{-t b_j}, i, k\right] + a\left[\frac{\left(e^{-t b_j} - 1\right) b_i}{b_j}, j, k\right] + ca\left[\frac{1 - e^{-t b_j}}{b_j}, k, i, j\right] +$$
$$ca\left[\frac{e^{-t b_j} - 1}{b_j}, j, i, k\right] + ca\left[\frac{b_i \left(1 - e^{-t b_j} - t b_j\right)}{b_j^2}, j, j, k\right] +$$
$$ca\left[\frac{e^{-2 t b_j} b_i \left(1 - e^{t b_j} - e^{t b_j}\left(e^{t b_j} - 2\right) t b_j\right)}{b_j^2}, k, j, k\right] + ca\left[\frac{e^{-2 t b_j}\left(e^{t b_j}\left(1 - t b_j\right) - 1\right)}{b_j},\right.$$
$$\left. k, i, k\right] + \delta a\left[\frac{b_i \left(-1 + e^{-t b_j} + t b_j\right) \epsilon_2}{b_j^2} + \frac{b_i \left(1 - e^{-2 t b_j} + \left(-1 - e^{-t b_j}\right) t b_j\right) \epsilon_2}{b_j^2}, j, k\right] +$$
$$\delta a\left[-\frac{\left(-1 + e^{-t b_j} + t b_j\right) \epsilon_2}{b_j} - \frac{\left(1 - e^{-2 t b_j} + \left(-1 - e^{-t b_j}\right) t b_j\right) \epsilon_2}{b_j}, i, k\right] +$$
$$\delta aa\left[\frac{2 e^{-t b_j} b_i (\text{Sinh}[t b_j] - t b_j)}{b_j^3}, j, k, j, k\right] + \delta aa\left[\frac{-1 + e^{-t b_j} + t b_j}{b_j^2}, i, j, j, k\right] +$$
$$\delta aa\left[-\frac{1 - e^{-2 t b_j} + \left(-1 - e^{-t b_j}\right) t b_j}{b_j^2}, i, k, j, k\right];$$

```
Ad[a[t_, j_, k_]][a[1, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
  a[ⅇ^(-t b_j), i, k] + a[((1 - ⅇ^(-t b_j)) b_i)/b_j, j, k] + ca[(2 ⅇ^(-t b_j) b_i (Sinh[t b_j] - t b_j))/b_j^2, k, j, k] +
   ca[(ⅇ^(-2 t b_j) (1 + ⅇ^(t b_j) (-1 + t b_j)))/b_j, k, i, k] + δa[-(ⅇ^(-2 t b_j) b_i (-1 + ⅇ^(t b_j) (1 - t b_j)) ϵ_2)/b_j^2, j, k] +
   δa[(ⅇ^(-2 t b_j) (-1 + ⅇ^(t b_j) (1 - t b_j)) ϵ_2)/b_j, i, k] + δaa[(2 ⅇ^(-t b_j) b_i (-Sinh[t b_j] + t b_j))/b_j^3,
    j, k, j, k] + δaa[(ⅇ^(-2 t b_j) (-1 + ⅇ^(t b_j) (1 - t b_j)))/b_j^2, i, k, j, k];
Ad[a[t_, j_, k_]][a[1, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[1, j, l] + ca[t, l, j, k] + ca[(ⅇ^(-t b_j) - 1)/b_j, k, j, l] + δaa[(1 - ⅇ^(-t b_j) - t b_j)/b_j^2, j, k, j, l];
Ad[a[t_, j_, k_]][a[1, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[ⅇ^(t b_j), k, l] + a[-((-1 + ⅇ^(t b_j)) b_k)/b_j, j, l] + ca[(-1 + ⅇ^(t b_j) (1 - t b_j))/b_j, k, k, l] +
   ca[(b_j - ⅇ^(-t b_j) b_j + b_k + ⅇ^(t b_j) (-1 + t b_j) b_k)/b_j^2, k, j, l] +
   ca[(b_j + b_k + t b_j b_k - ⅇ^(t b_j) (b_j + b_k))/b_j^2, l, j, k] + δaa[(1 + ⅇ^(t b_j) (-1 + t b_j))/b_j^2, j, k, k, l] +
   δaa[1/b_j^3 ⅇ^(-t b_j) (b_j + ⅇ^(2 t b_j) (b_j + (2 - t b_j) b_k) - ⅇ^(t b_j) (2 b_k + b_j (2 + t b_k))), j, k, j, l];
Ad[a[t_, j_, k_]][c[1, i_]] /; FreeQ[t, b_] ∧ ({j, k} ⋂ {i} === {}) := c[1, i];
Ad[a[t_, j_, k_]][c[1, j_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[1, j] + c[1 - ⅇ^(-t b_j), k] + δa[(ⅇ^(-t b_j) - 1)/b_j, j, k];
Ad[a[t_, j_, k_]][c[1, k_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[ⅇ^(-t b_j), k] + δa[(1 - ⅇ^(-t b_j))/b_j, j, k];

Ad[x_β | x_c | x_δa | x_ca | x_δaa][y_] := y + B[x, y];

Ad[x_][a[f_, i_, j_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][a[1, i, j]];
Ad[x_][c[f_, i_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][c[1, i]];
Ad[x_][δa[f_, j_, k_]] := δ ** (β[f] ** Ad[x][a[1, j, k]]);
Ad[x_][ca[f_, i_, j_, k_]] := Ad[x][c[f, i]] ** Ad[x][a[1, j, k]];
Ad[x_][δaa[f_, i_, j_, k_, l_]] := Ad[x][δa[f, i, j]] ** Ad[x][a[1, k, l]];
Ad[x_][y_Plus] := Ad[x] /@ y;

Ad::ndef = "Ad[`1` is not defined on `2`.";
Ad[x_][y_] := (Message[Ad::ndef, x, y]; Undefined);
```

AutoAd - Ad tests

```
Module[{t1, t2},
   {t1 = S[AutoAd[a[t, j, k]][#]],
    S[Ad[a[t, j, k]][#] - t1]}
  ] & @ a[1, i, k]
```

$$\left\{a\left[e^{-t b_j}, i, k\right] + a\left[\frac{\left(1 - e^{-t b_j}\right) b_i}{b_j}, j, k\right] + \mathrm{ca}\left[\frac{e^{-2 t b_j} b_i \left(-1 + e^{2 t b_j} - 2 e^{t b_j} t b_j\right)}{b_j^2}, k, j, k\right] + \mathrm{ca}\left[\frac{e^{-2 t b_j} \left(1 - e^{t b_j} + e^{t b_j} t b_j\right)}{b_j}, k, i, k\right] + \delta a\left[-\frac{e^{-2 t b_j} b_i \left(1 - e^{t b_j} + e^{t b_j} t b_j\right)}{b_j^2}, j, k\right] + \delta a\left[\frac{e^{-2 t b_j} \left(1 - e^{t b_j} + e^{t b_j} t b_j\right)}{b_j}, i, k\right] + \delta \mathrm{aa}\left[-\frac{e^{-2 t b_j} b_i \left(-1 + e^{2 t b_j} - 2 e^{t b_j} t b_j\right)}{b_j^3}, j, k, j, k\right] + \delta \mathrm{aa}\left[\frac{e^{-2 t b_j} \left(-1 + e^{t b_j} - e^{t b_j} t b_j\right)}{b_j^2}, i, k, j, k\right], 0\right\}$$

```
AdTests[a[t, j, k]] = {β[f[bj, bk]], a[1, j, k], a[1, n, i], a[1, j, j], a[1, k, k],
   c[1, i], c[1, j], c[1, k], a[1, j, l], a[1, i, j], a[1, i, k], a[1, k, l]};
```

```
S[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ Take[AdTests[a[t, j, k]], All]
```

```
$Aborted
```

The semi group properties

```
Module[{t1, t2, t3, t4},
    t1 = Ad[a[t, j, k]][#] /.
      (h : (β | a | c | δa | ca | δaa))[c_, r___] :> h[SeriesCoefficient[c, {t, 0, 1}], r];
    t2 = B[a[1, j, k], #];
    t3 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
    t4 = # // Ad[a[t + s, j, k]];
    # → S[{t1 == t2, t3 - t4}]
  ] & /@ AdTests[a[t, j, k]] // ColumnForm
```

```
β[f[bj, bk]] → {True, 0}
a[1, j, k] → {True, 0}
a[1, n, i] → {True, 0}
a[1, j, j] → {True, 0}
a[1, k, k] → {True, 0}
c[1, i] → {True, 0}
c[1, j] → {True, 0}
c[1, k] → {True, 0}
a[1, j, l] → {True, 0}
a[1, i, j] → {True, 0}
a[1, i, k] → {True, 0}
a[1, k, l] → {True, 0}
```

R

```
Switch[10,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t bi, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[bi, bj]]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bj] + bi f1[bj]]],
  4, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[c[f3[bi, bj], i]] // Ad[c[f4[bi, bj], j]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f5[bi], i, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f6[bi, bj], j, i, j]],
  7, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[ca[f5[bi], i, i, j]] // Ad[ca[f6[bi, bj], j, i, j]],
  8, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[δa[f2[bi, bj], j, i]],
  9, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[ca[f7[bi, bj], i, j, i]] // Ad[ca[f8[bi, bj], j, j, i]],
  10, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f8[bi, bj], j, j, i]]
 ];
VerifyR3[expr_] := Module[{lhs, rhs}, {
   lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // S;
   rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // S;
   S[lhs - rhs] == 0
  }]
```

Verifying R3

```
VerifyR3 /@ {β[f[b1, b2, b3, b4]], c[f[b1, b2, b3, b4], 1], c[f[b1, b2, b3, b4], 2],
  c[f[b1, b2, b3, b4], 3], δa[f[b1, b2, b3, b4], 1, 2], δa[f[b1, b2, b3, b4], 1, 3],
  δa[f[b1, b2, b3, b4], 2, 3], a[f[b1, b2, b3, b4], 1, 4], a[f, 2, 4]}
```

$$\begin{aligned}
&\{\{\mathrm{True}\}, \{\mathrm{True}\}, \{\mathrm{True}\}, \{\mathrm{True}\}, \{\mathrm{True}\}, \{\mathrm{True}\}, \{\mathrm{True}\},\\
&\Big\{\mathrm{ca}\big[e^{-b_2}\left(-1+e^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(f8[b_1,b_2]\,b_2 - f8[b_1,b_3]\,b_3), 4, 1, 3\big] +\\
&\quad \mathrm{ca}\big[-\tfrac{1}{b_2} e^{-b_2}\left(-1+e^{b_2}\right) f[b_1,b_2,b_3,b_4]\,b_1\,(f8[b_1,b_2]\,b_2 - f8[b_1,b_3]\,b_3), 4, 2, 3\big] +\\
&\quad \delta\mathrm{aa}\Big[-\frac{\left(-1+e^{b_2}\right) f[b_1,b_2,b_3,b_4]\,f8[b_1,b_3]\,b_1}{b_2}, 2, 3, 3, 4\Big] +\\
&\quad \delta\mathrm{aa}\big[-\tfrac{1}{b_2} e^{-b_2}\left(-1+e^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(f8[b_1,b_2]\,b_2 - f8[b_1,b_3]\,b_3), 1, 3, 2, 4\big] +\\
&\quad \delta\mathrm{aa}\big[\tfrac{1}{b_2^2} e^{-b_2}\left(-1+e^{b_2}\right) f[b_1,b_2,b_3,b_4]\,b_1\left(f8[b_1,b_2]\,b_2 + \left(-1+e^{b_2}\right) f8[b_1,b_3]\,b_3\right),\\
&\quad\; 2, 3, 2, 4\big] == 0\Big\}, \Big\{\mathrm{ca}\big[\left(-1+e^{b_1}\right) f\,(f8[b_1,b_3] - f8[b_2,b_3])\,b_2, 3, 3, 4\big] +\\
&\quad \mathrm{ca}\big[e^{-b_2}\left(-1+e^{b_1}\right) f\left(\left(-2+e^{b_2}\right) f8[b_1,b_2]\,b_2 - \left(-1+e^{b_2}\right) f8[b_1,b_3]\,b_3\right), 4, 2, 3\big] +\\
&\quad \mathrm{ca}\big[-\tfrac{1}{b_1} e^{-b_2}\left(-1+e^{b_1}\right) f\,b_2\left(\left(-2+e^{b_2}\right) f8[b_1,b_2]\,b_2 - \left(-1+e^{b_2}\right) f8[b_1,b_3]\,b_3\right), 4, 1, 3\big] +\\
&\quad \mathrm{ca}\big[-\tfrac{1}{b_1^2} e^{-b_2} f\,b_2\,\big(2\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right) +\\
&\qquad\quad b_1\left(-\left(1+e^{b_1}\right)\left(-1+e^{b_2}\right) + e^{b_2}\left(-1+e^{b_1}\right) f8[b_1,b_3]\,b_3\right)\big), 3, 1, 4\big] + \mathrm{ca}\big[\\
&\quad\; \tfrac{1}{b_1} e^{-b_2} f\left(2\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right) + b_1\left(-\left(1+e^{b_1}\right)\left(-1+e^{b_2}\right) + e^{b_2}\left(-1+e^{b_1}\right) f8[b_2,b_3]\,b_3\right)\right),\\
&\quad\; 3, 2, 4\big] + \delta\mathrm{aa}\big[\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right) f\,f8[b_1,b_3], 2, 3, 3, 4\big] +\\
&\quad \delta\mathrm{aa}\Big[\frac{\left(-1+e^{b_1}\right) f\,f8[b_2,b_3]\,b_2}{b_1}, 1, 3, 3, 4\Big] +\\
&\quad \delta\mathrm{aa}\Big[-\frac{e^{-b_2}\left(-1+e^{b_2}\right) f\left(2 - 2e^{b_1} + \left(1+e^{b_1}\right) b_1\right) b_2}{b_1^3}, 1, 3, 1, 4\Big] +\\
&\quad \delta\mathrm{aa}\big[-\tfrac{1}{b_2} e^{-b_2}\left(-1+e^{b_1}\right) f\left(\left(-2+e^{b_2}\right) f8[b_1,b_2]\,b_2 + \left(-1+e^{b_2}\right)^2 f8[b_1,b_3]\,b_3\right),\\
&\quad\; 2, 3, 2, 4\big] + \delta\mathrm{aa}\big[\tfrac{1}{b_1^2} e^{-b_2} f\,\big(-2\left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right) +\\
&\qquad\quad b_1\big(-1 - e^{b_1} + e^{b_2} + e^{b_1+b_2} + \left(-1+e^{b_1}\right)\left(-2+e^{b_2}\right) f8[b_1,b_2]\,b_2 - \left(-1+e^{b_1}\right)\left(-1+e^{b_2}\right)\\
&\qquad\qquad f8[b_1,b_3]\,b_3 + e^{b_2} f8[b_2,b_3]\,b_3 - e^{b_1+b_2} f8[b_2,b_3]\,b_3\big)\big), 1, 3, 2, 4\big] == 0\Big\}\Big\}
\end{aligned}$$

```
FormalBasis[2, f]
```

$\{\beta[f[b_1, b_2]], a[f[b_1, b_2], 1, 1], a[f[b_1, b_2], 1, 2], a[f[b_1, b_2], 2, 1], a[f[b_1, b_2], 2, 2], c[f[b_1, b_2], 1], c[f[b_1, b_2], 2], \delta a[f[b_1, b_2], 1, 1], \delta a[f[b_1, b_2], 1, 2], \delta a[f[b_1, b_2], 2, 1], \delta a[f[b_1, b_2], 2, 2], ca[f[b_1, b_2], 1, 1, 1], ca[f[b_1, b_2], 1, 1, 2], ca[f[b_1, b_2], 1, 2, 1], ca[f[b_1, b_2], 1, 2, 2], ca[f[b_1, b_2], 2, 1, 1], ca[f[b_1, b_2], 2, 1, 2], ca[f[b_1, b_2], 2, 2, 1], ca[f[b_1, b_2], 2, 2, 2], \delta aa[f[b_1, b_2], 1, 1, 1, 1], \delta aa[f[b_1, b_2], 1, 1, 1, 2], \delta aa[f[b_1, b_2], 1, 1, 2, 1], \delta aa[f[b_1, b_2], 1, 1, 2, 2], \delta aa[f[b_1, b_2], 1, 2, 1, 2], \delta aa[f[b_1, b_2], 1, 2, 2, 2], \delta aa[f[b_1, b_2], 2, 1, 2, 1], \delta aa[f[b_1, b_2], 2, 1, 2, 2], \delta aa[f[b_1, b_2], 2, 2, 2, 2]\}$

```
FormalBasis[2, f] // Length
```

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