

Pensieve header: One-Co computations in the abc presentation; continues pensieve://2015-07/.

The bracket

On the elements β , a , c , δa , ca , δaa .

Generalities

```

DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];

Simp[expr_] := Simplify[expr];
S[ $\beta$ [f_]] :=  $\beta$ [Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[ $\delta a$ [f_, i_, j_]] :=  $\delta a$ [Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[ $\delta aa$ [f_, i_, j_, k_, l_]] :=  $\delta aa$ [Simp[f], i, j, k, l];
S[expr_] := expr /. ( $\lambda_\beta$  |  $\lambda_a$  |  $\lambda_{\delta a}$  |  $\lambda_c$  |  $\lambda_{ca}$  |  $\lambda_{\delta aa}$ )  $\rightarrow$  S[ $\lambda$ ];

 $\beta$ [0] := 0;
 $\beta$  /:  $\beta$ [f_] +  $\beta$ [g_] :=  $\beta$ [f+g] // S;
 $\beta$  /: g_ *  $\beta$ [f_] :=  $\beta$ [gf] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f+g, j, k] // S;
a /: g_ * a[f_, j_, k_] := a[gf, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f+g, j] // S;
c /: g_ * c[f_, j_] := c[gf, j] // S;
 $\delta a$ [0, _, _] := 0;
 $\delta a$  /:  $\delta a$ [f_, j_, k_] +  $\delta a$ [g_, j_, k_] :=  $\delta a$ [f+g, j, k] // S;
 $\delta a$  /: g_ *  $\delta a$ [f_, j_, k_] :=  $\delta a$ [gf, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f+g, j, k, l] // S;
ca /: g_ * ca[f_, j_, k_, l_] := ca[gf, j, k, l] // S;
 $\delta aa$ [0, _, _, _, _] := 0;
 $\delta aa$  /:  $\delta aa$ [f_, i_, j_, k_, l_] +  $\delta aa$ [g_, i_, j_, k_, l_] :=
   $\delta aa$ [f+g, i, j, k, l] // S;
 $\delta aa$  /: g_ *  $\delta aa$ [f_, i_, j_, k_, l_] :=  $\delta aa$ [gf, i, j, k, l] // S;

```

δ_{aa} relations

VS = False;

"First sort tails then sort heads"

Locality - sorts tails when supports are disjoint:

```
S[ $\delta_{aa}[f_, i_, j_, k_, l_] /; (\{i, j\} \cap \{k, l\} == \{\}) \wedge !OQ[i, k] := ($ 
  If[VS, Print["Locality on ",  $\delta_{aa}[f, i, j, k, l]$ ]];
   $\delta_{aa}[f, k, l, i, j]$  // S
);
```

Commute Heads - sorts tails when the heads are the same:

```
S[ $\delta_{aa}[f_, i_, k_, j_, k_] /; DQ[i, j, k] \wedge !OQ[i, j] := ($ 
  If[VS, Print["Commute heads on ",  $\delta_{aa}[f, i, k, j, k]$ ]];
  S[Expand[
     $\delta_{aa}[f, j, k, i, k] + e_2 (\delta_a[b_i f, j, k] - \delta_a[b_j f, i, k])$ 
  ]]
);
```

Commute Head/Tail - sorts tails:

```
S[ $\delta_{aa}[f_, i_, j_, k_, i_] /; DQ[i, j, k] \wedge !OQ[i, k] := ($ 
  If[VS, Print["Commute head/tail on ",  $\delta_{aa}[f, i, j, k, i]$ ]];
  S[
     $\delta_{aa}[f, k, i, i, j] + \delta_{aa}[f, k, j, i, j] - \delta_{aa}[f, i, j, k, j]$ 
  ]
);
```

2113 Swinging - sorts tails:

```
 $\delta_{aa}[f, j, i, ii, k]$  // S
ca[-fbii, i, j, k] + ca[fbii, k, j, i] +
ca[-fbj, k, ii, i] + ca[fbj, i, ii, k] +  $\delta_{aa}[f, ii, i, j, k]$ 

S[ $\delta_{aa}[f_, j_, i_, i_, k_] /; DQ[i, j, k] \wedge OQ[i, j, k] := ($ 
  If[VS, Print["2113 swinging on ",  $\delta_{aa}[f, j, i, i, k]$ ]];
  S[ca[-fbi, i, j, k] + ca[fbi, k, j, i] +
    c[-fbj, k] ** adown[1, i] + ca[fbj, i, i, k] + adown[f, i] **  $\delta_a[1, j, k]$ ]
];
```

$\delta_{aa}[f, j, i, i, k]$ // S

```
c[-fbibj ∈1, k] + c[-fbi, i, j, k] + c[fbi, k, j, i] +
ca[-fbj, k, i, i] + ca[fbj, i, i, k] +  $\delta_a[fb_i \in_1, j, k]$  +  $\delta_{aa}[f, i, i, j, k]$ 
```

3112 Swinging - sorts tails:

$\delta_{aa}[f, k, i, ii, j]$ // S

Locality on $\delta_{aa}[f, k, i, ii, j]$

Standard swinging on $\delta_{aa}[f, ii, j, k, i]$

$ca[-fb_{ii}, i, k, j] + ca[fb_{ii}, j, k, i] +$
 $ca[-fb_k, j, ii, i] + ca[fb_k, i, ii, j] + \delta_{aa}[f, ii, i, k, j]$

$S[\delta_{aa}[f_, k_, i_, ii_, j_]]$ /; $DQ[i, j, k] \wedge OQ[i, j, k] := ($
 If[VS, Print["3112 swinging on ", $\delta_{aa}[f, k, i, ii, j]$]];
 $S[ca[-fb_i, i, k, j] + ca[fb_i, j, k, i] +$
 $c[-fb_k, j] ** adown[1, i] + ca[fb_k, i, ii, j] + adown[1, i] ** \delta_a[f, k, j]$
 $);$

$\delta_{aa}[f, k, i, i, j]$ // S

3112 swinging on $\delta_{aa}[f, k, i, i, j]$

$c[-fb_i b_k \in_1, j] + ca[-fb_i, i, k, j] + ca[fb_i, j, k, i] +$
 $ca[-fb_k, j, i, i] + ca[fb_k, i, i, j] + \delta_a[fb_i \in_1, k, j] + \delta_{aa}[f, i, i, k, j]$

Standard Swinging - sorts heads, if support is 4 strands:

$S[\delta_{aa}[f_, i_, j_, k_, l_]]$ /; $DQ[i, j, k, l] \wedge OQ[i, k] \wedge !OQ[j, l] := ($
 If[VS, Print["Standard swinging on ", $\delta_{aa}[f, i, j, k, l]$]];
 $S[\delta_{aa}[f, i, l, k, j] + ca[b_k f, l, i, j] -$
 $ca[b_i f, l, k, j] - ca[b_k f, j, i, l] + ca[b_i f, j, k, l]$
 $);$

Tails Commute - sorts heads when the tails are the same:

$S[\delta_{aa}[f_, i_, j_, i_, l_]]$ /; $DQ[i, j, l] \wedge !OQ[j, l] := ($
 If[VS, Print["Tails commute on ", $\delta_{aa}[f, i, j, i, l]$]];
 $\delta_{aa}[f, i, l, i, j]$ // S
 $);$

1321 Swinging - sorts heads:

$\delta_{aa}[f, i, k, j, ii]$ // S

Standard swinging on $\delta_{aa}[f, i, k, j, ii]$

$ca[-fb_i, ii, j, k] + ca[fb_i, k, j, ii] +$
 $ca[-fb_j, k, i, ii] + ca[fb_j, ii, i, k] + \delta_{aa}[f, i, ii, j, k]$

$S[\delta_{aa}[f_, i_, k_, j_, ii_]]$ /; $DQ[i, j, k] \wedge OQ[i, j, k] := ($
 If[VS, Print["1321 swinging on ", $\delta_{aa}[f, i, k, j, ii]$]];
 $S[ca[-fb_i, i, j, k] + ca[fb_i, k, j, ii] +$
 $ca[-fb_j, k, i, ii] + ac[fb_j, i, k, ii] + \delta_{aa}[f, i, ii, j, k]$
 $);$

1322 Swinging - sorts heads, but breaks tails:

```

 $\delta_{aa}[f, i, k, j, jj]$  // S
Standard swinging on  $\delta_{aa}[f, i, k, j, jj]$ 
ca[-fbi, jj, j, k] + ca[fbi, k, j, jj] +
  ca[-fbj, k, i, jj] + ca[fbj, jj, i, k] +  $\delta_{aa}[f, i, jj, j, k]$ 

S[ $\delta_{aa}[f, i, k, j, jj]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["1322 swinging on ",  $\delta_{aa}[f, i, k, j, jj]$ ]];
  S[ac[-fbi, j, k, j] + ca[fbi, k, j, j] +
    ca[-fbj, k, i, j] + ca[fbj, j, i, k] +  $\delta_{aa}[f, j, k, i, j]$ ]
);

```

1332 Swinging - sorts heads:

```

 $\delta_{aa}[f, i, k, kk, j]$  // S
Standard swinging on  $\delta_{aa}[f, i, k, kk, j]$ 
ca[-fbi, j, kk, k] + ca[fbi, k, kk, j] +
  ca[-fbkk, k, i, j] + ca[fbkk, j, i, k] +  $\delta_{aa}[f, i, j, kk, k]$ 

S[ $\delta_{aa}[f, i, k, kk, j]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["1332 swinging on ",  $\delta_{aa}[f, i, k, kk, j]$ ]];
  S[ca[-fbi, j, k, k] + ca[fbi, k, k, j] +
    ca[-fbk, k, i, j] + ca[fbk, j, i, k] +  $\delta_a[f, i, j]$  ** adown[1, k]]
);

```

```

 $\delta_{aa}[f, i, k, k, j]$  // S
1332 swinging on  $\delta_{aa}[f, i, k, k, j]$ 
ca[-fbi, j, k, k] + ca[fbi, k, k, j] + ca[-fbk, k, i, j] +
  ca[fbk, j, i, k] +  $\delta_a[fb_k \in_1, i, j]$  +  $\delta_{aa}[f, i, j, k, k]$ 

```

1231 Swinging - sorts heads:

```

 $\delta_{aa}[f, i, j, k, ii]$  // S
Standard swinging on  $\delta_{aa}[f, i, j, k, ii]$ 
ca[-fbi, ii, k, j] + ca[fbi, j, k, ii] +
  ca[-fbk, j, i, ii] + ca[fbk, ii, i, j] +  $\delta_{aa}[f, i, ii, k, j]$ 

S[ $\delta_{aa}[f, i, j, k, ii]$ ] /; DQ[i, j, k]  $\wedge$  OQ[i, j, k] := (
  If[VS, Print["1231 swinging on ",  $\delta_{aa}[f, i, j, k, ii]$ ]];
  S[ac[-fbi, k, j, i] + ca[fbi, j, k, i] +
    ca[-fbk, j, i, i] + ac[fbk, i, j, i] +  $\delta_{aa}[f, i, i, k, j]$ ]
);

```

```

 $\delta_{aa}[f, i, j, k, i]$  // S
1231 swinging on  $\delta_{aa}[f, i, j, k, i]$ 
c[f bi bk, j] + ca[-f bi, i, k, j] + ca[f bi, j, k, i] +
ca[-f bk, j, i, i] + ca[f bk, i, i, j] +  $\delta a[-f b_k, i, j]$  +  $\delta_{aa}[f, i, i, k, j]$ 

```

NonCommutativeMultiply

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_ , 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;

 $\beta[f_] ** a[g_, j_, k_] := a[fg, j, k];$ 
 $\beta[f_] ** c[g_, j_] := c[fg, j];$ 
 $c[g_, j_] ** \beta[f_] := c[fg, j];$ 
 $\beta[f_] ** \delta a[g_, j_, k_] := \delta a[fg, j, k];$ 
 $\beta[f_] ** ca[g_, i_, j_, k_] := ca[fg, i, j, k];$ 
 $ca[g_, i_, j_, k_] ** \beta[f_] := ca[fg, i, j, k];$ 
 $\beta[f_] ** \delta_{aa}[g_, i_, j_, k_, l_] := \delta_{aa}[fg, i, j, k, l];$ 
 $\delta a[g_, j_, k_] ** \beta[f_] := \delta a[fg, j, k];$ 
 $\delta ** a[f_, i_, j_] := \delta a[f, i, j];$ 
 $c[f_, i_] ** a[g_, j_, k_] := ca[fg, i, j, k];$ 
 $a[f_, i_, j_] ** \delta a[g_, k_, l_] := \delta_{aa}[fg, i, j, k, l];$ 
 $\delta a[f_, i_, j_] ** a[g_, k_, l_] := \delta_{aa}[fg, i, j, k, l];$ 

 $\delta ** _c = 0;$ 
 $\delta ** _\delta a = 0;$ 
 $\delta ** _ca = 0;$ 
 $\delta ** _\delta_{aa} = 0;$ 
 $_c ** _c = 0;$ 
 $_c ** _\delta a = _\delta a ** _c = 0;$ 
 $_c ** _ca = _ca ** _c = 0;$ 
 $_c ** _\delta_{aa} = _\delta_{aa} ** _c = 0;$ 
 $_\delta a ** _\delta a = 0;$ 
 $_\delta a ** _\delta_{aa} = _\delta_{aa} ** _\delta a = 0;$ 
 $_\delta a ** _ca = _ca ** _\delta a = 0;$ 

NonCommutativeMultiply::ndef =
"NonCommutativeMultiply is not defined on {\`1`,\`2`}."
NonCommutativeMultiply[x_, y_] :=
(Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
NonCommutativeMultiply is not defined on {\`1`,\`2`}.

```

Bracket Generalities

```

B[0, _] = 0; B[_ , 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[# , y] & /@ x;
B[x_, y_Plus] := B[x , #] & /@ y;

```

The γ shortcuts

```

 $\gamma[f_, j_, k_] := \delta a[f, j, k] - c[b_j f, k] // S;$ 
 $\gamma[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;$ 
ac[f_, j_, k_, l_] := ca[f, l, j, k] + B[a[l, j, k], c[f, l]];
adown[f_, j_] := a[f, j, j] +  $\beta$ [ $\epsilon_1 f b_j$ ] + c[ $\epsilon_1 f, j$ ];

```

Fundamental Brackets

a- β , a-c, a-a, AS

```

B[a[j_, k_],  $\beta$ [g_]] :=  $\gamma[\partial_{b_j} g - \partial_{b_k} g, j, k];$ 
B[ $\beta$ [g_], a[j_, k_]] := -B[a[j, k],  $\beta$ [g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k}  $\cap$  {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] :=  $\gamma[1, j, k, l] // S;$ 
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] -  $\gamma[1, j, k, l] // S;$ 
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
(* backie *) B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[ $\epsilon_5 b_j, k, j$ ] - a[ $\epsilon_5 b_k, j, k$ ] + a[ $\epsilon_5 b_j, k, k$ ] - a[ $\epsilon_5 b_k, j, j$ ] + ca[ $\epsilon_4, k, k, j$ ] -
  ca[ $\epsilon_4, j, j, k$ ] + ca[ $\epsilon_4, k, j, j$ ] - ca[ $\epsilon_4, j, k, k$ ] +  $\gamma[\epsilon_4, j, k] - \gamma[\epsilon_4, k, j];$ 
(* [tail, selfie] *) B[a[j_, k_], a[j_, j_]] /; DQ[j, k] :=  $\gamma[-\epsilon_6, j, k] // S;$ 
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
(* [head, selfie] *) B[a[j_, k_], a[k_, k_]] /; DQ[j, k] :=  $\gamma[-\epsilon_7, j, k] // S;$ 
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[k, k]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] :=  $\gamma[-fg, j, k];$ 
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] :=  $\gamma[fg, j, k];$ 
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k}  $\cap$  {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];

```

Vanishing brackets

```

B[_ $\beta$ , _ $\beta$  |  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa] = 0;
B[_ $\beta$  |  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa, _ $\beta$ ] = 0;
B[ $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa,  $\delta$  | _c | _ $\delta$ a | _ca | _ $\delta$ aa] = 0;

```

Composite Brackets

```

B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[l, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[l, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[l, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[l, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];

B::ndef = "B is not defined on {\`1`,\`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
B is not defined on {\`1`,\`2`}.

```

Testing Jacobi and Anti-Symmetry

```

FormalBasis[n_, f_] := Module[{ff},
  ff = f@@Table[b_i, {i, n}];
  Flatten@{
     $\beta$ [ff],
    Table[a[ff, i, j], {i, n}, {j, n}],
    Table[c[ff, i], {i, n}],
    Table[ $\delta a$ [ff, i, j], {i, n}, {j, n}],
    Table[ca[ff, i, j, k], {i, n}, {j, n}, {k, n}],
    Table[ $\delta aa$ [ff, i, j, k, l], {i, n}, {j, n}, {k, i, n}, {l, j, n}]
  } /. 1[___]  $\rightarrow$  1
];

FormalPlusBasis[n_, f_] := Module[{ff},
  ff = f@@Table[b_i, {i, n}];
  Flatten@{
     $\beta$ [ff],
    Table[a[ff, i, j], {i, n-1}, {j, i+1, n}],
    Table[c[ff, i], {i, n}],
    Table[ $\delta a$ [ff, i, j], {i, n-1}, {j, i+1, n}],
    Table[ca[ff, i, j, k], {i, n}, {j, n-1}, {k, j+1, n}],
    Table[ $\delta aa$ [ff, i, j, k, l], {i, n-1}, {j, i+1, n}, {k, n-1}, {l, k+1, n}]
  } /. 1[___]  $\rightarrow$  1
];

VS = False;
AS[x1_, x2_] := Module[{as},
  as = B[x1, x2] + B[x2, x1] // S;
  If[as === 0, as, {x1, x2}  $\rightarrow$  as]
];

DeleteCases[Flatten[Outer[
  AS,
  FormalPlusBasis[3, f],
  FormalPlusBasis[3, g]
]], 0]
{}

```



```

AS[x1_, x2_] := Module[{as},
  as = B[x1, x2] + B[x2, x1] // S;
  If[as === 0, as, {x1, x2} → as]
];
DeleteCases[Flatten[Outer[
  AS,
  FormalBasis[3, f],
  FormalBasis[3, g]
]], 0]
{}

Jacobi[x1_, x2_, x3_] := Module[{Jac},
  Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
  If[Jac === 0, Jac, {x1, x2, x3} → Jac]
];

JacPlusErrors = DeleteCases[
  bas1 = FormalPlusBasis[4, f];
  bas2 = FormalPlusBasis[4, g];
  bas3 = FormalPlusBasis[4, h];
  Flatten[
    Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]],
      {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
{}

JacPlusErrors // Length
0

```

```
JacErrors = DeleteCases [
  bas1 = FormalBasis[4, f];
  bas2 = FormalBasis[4, g];
  bas3 = FormalBasis[4, h];
  Flatten[
    Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]],
      {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

$\{ \{ a[f[b_1, b_2, b_3, b_4], 1, 1], a[g[b_1, b_2, b_3, b_4], 1, 2], a[h[b_1, b_2, b_3, b_4], 2, 1] \} \rightarrow$
 $c[-2 f[b_1, b_2, b_3, b_4] g[b_1, b_2, b_3, b_4] h[b_1, b_2, b_3, b_4] b_1 b_2, 1] +$
 $c[2 f[b_1, b_2, b_3, b_4] g[\dots 1 \dots] \dots 1 \dots b_1 b_2, 2] + \delta a[\dots 1 \dots] +$
 $\delta a[2 f[b_1, b_2, b_3, b_4] g[b_1, b_2, b_3, b_4] h[b_1, b_2, b_3, b_4] b_2, 1, 1],$
 $\dots 1520 \dots, \dots 1 \dots \}$

[large output](#) | [show less](#) | [show more](#) | [show all](#) | [set size limit...](#)

JacErrors // Length

1522

The Adjoint action

AutoAd

```

AutoAd[x_][y_] :=
Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
  pows = NestList[B[x, #] &, y, 20];
  Print["pows computed for ", {x, y}, "..."];
  states = Union[
    Cases[pows, s_β | s_a | s_c | s_δa | s_ca | s_δaa > ReplacePart[s, 1 → _], ∞]];
  Sum[
    seq = Cases[{-#}, states[[i]], ∞] & /@ pows;
    seq = Replace[seq, {_{f_, ___}} > f, {} → 0}, {1}];
    Print["seq computed... ", states[[i]], " is ", i, "/", Length@states];
    dseq = Drop[seq, sh];
    If[Union[Length[MonomialList[{-#}] & /@ dseq] === {1} &
      Union[Length[FactorTermsList[{-#}] & /@ dseq] === {2},
      sf1 = FindSequenceFunction[FactorTermsList[{-#][1] & /@ dseq];
      sf2 = FindSequenceFunction[FactorTermsList[{-#][2] & /@ dseq];
      Print["sf1: ", sf1, " sf2: ", sf2];
      sf = (sf1[{-#] sf2[{-#] &),
      (*Else*) sf = FindSequenceFunction[dseq,
        FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
          "Polynomial", "RationalFunction", "HypergeometricTerm"}];
      Print["sf: ", sf];
    ];
    ReplacePart[states[[i], 1 → Simplify[
      
$$\sum_{n=0}^{sh-1} \frac{seq[[n+1]]}{n!} + \sum_{n=sh}^{\infty} \frac{sf[n+1-sh]}{n!}$$

    ]],
      {i, Length@states}];
  ];
  (* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)
AutoAd[a[t, j, k]][β[f[b_j, b_k]]]

```

pows computed for {a[t, j, k], $\beta[f[b_j, b_k]]$ }...
 seq computed... c[_ , k] is 1/3
 sf: $-t^4 b_j^4 (-tb_j)^{\#1} (f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k]) \&$
 seq computed... $\beta[_]$ is 2/3
 sf1: 1 & sf2: 0 &
 seq computed... $\delta a[_ , j, k]$ is 3/3
 sf: $t^4 b_j^3 (-tb_j)^{\#1} (f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k]) \&$
 $c[e^{-tb_j} (-1 + e^{tb_j}) (f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k]), k] +$
 $\beta[f[b_j, b_k]] + \delta a[-\frac{e^{-tb_j} (-1 + e^{tb_j}) (f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k])}{b_j}, j, k]$

Ad

$\text{Ad}[a[t_, j_, k_]][\beta[f_]] /; \text{FreeQ}[t, b_] :=$
 $\beta[f] + c[(1 - e^{-tb_j}) (\partial_{b_k} f - \partial_{b_j} f), k] + \delta a[\frac{(e^{-tb_j} - 1) (\partial_{b_k} f - \partial_{b_j} f)}{b_j}, j, k];$
 $\text{Ad}[a[t_, j_, k_]][a[1, j_, k_]] /; \text{FreeQ}[t, b_] := a[1, j, k];$
 $\text{Ad}[a[t_, j_, k_]][a[1, n_, i_]] /;$
 $\text{FreeQ}[t, b_] \wedge (\{j, k\} \cap \{n, i\} == \{\}) := a[1, n, i];$
 $\text{Ad}[a[t_, j_, k_]][a[1, i_, j_]] /; \text{DQ}[i, j, k] \wedge \text{FreeQ}[t, b_] :=$
 $a[1, i, j] + a[1 - e^{-tb_j}, i, k] + a[\frac{(e^{-tb_j} - 1) b_i}{b_j}, j, k] + ca[\frac{1 - e^{-tb_j}}{b_j}, k, i, j] +$
 $ca[\frac{e^{-tb_j} - 1}{b_j}, j, i, k] + ca[\frac{b_i (1 - e^{-tb_j} - tb_j)}{b_j^2}, j, j, k] +$
 $ca[\frac{e^{-2tb_j} b_i (1 - e^{tb_j} - e^{-tb_j} (e^{tb_j} - 2) tb_j)}{b_j^2}, k, j, k] + ca[\frac{e^{-2tb_j} (e^{tb_j} (1 - tb_j) - 1)}{b_j},$
 $k, i, k] + \delta a[\frac{b_i (-1 + e^{-tb_j} + tb_j) \epsilon_2}{b_j^2} + \frac{b_i (1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j) \epsilon_2}{b_j^2}, j, k] +$
 $\delta a[-\frac{(-1 + e^{-tb_j} + tb_j) \epsilon_2}{b_j} - \frac{(1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j) \epsilon_2}{b_j}, i, k] +$
 $\delta aa[\frac{2 e^{-tb_j} b_i (\text{Sinh}[tb_j] - tb_j)}{b_j^3}, j, k, j, k] + \delta aa[\frac{-1 + e^{-tb_j} + tb_j}{b_j^2}, i, j, j, k] +$
 $\delta aa[-\frac{1 - e^{-2tb_j} + (-1 - e^{-tb_j}) tb_j}{b_j^2}, i, k, j, k];$
 $\text{Ad}[a[t_, j_, k_]][a[1, i_, k_]] /; \text{DQ}[i, j, k] \wedge \text{FreeQ}[t, b_] :=$
 $a[e^{-tb_j}, i, k] + a[\frac{(1 - e^{-tb_j}) b_i}{b_j}, j, k] + ca[\frac{2 e^{-tb_j} b_i (\text{Sinh}[tb_j] - tb_j)}{b_j^2}, k, j, k] +$
 $ca[\frac{e^{-2tb_j} (1 + e^{tb_j} (-1 + tb_j))}{b_j}, k, i, k] + \delta a[-\frac{e^{-2tb_j} b_i (-1 + e^{tb_j} (1 - tb_j)) \epsilon_2}{b_j^2}, j, k] +$

```


$$\delta a \left[ \frac{e^{-2tb_j} (-1 + e^{tb_j} (1 - tb_j))}{b_j}, i, k \right] + \delta aa \left[ \frac{2 e^{-tb_j} b_i (-\text{Sinh}[tb_j] + tb_j)}{b_j^3}, j, k, j, k \right] + \delta aa \left[ \frac{e^{-2tb_j} (-1 + e^{tb_j} (1 - tb_j))}{b_j^2}, i, k, j, k \right];$$

Ad[a[t_, j_, k_]] [a[1, j_, l_]] /; DQ[j, k, l] & FreeQ[t, b_] :=
a[1, j, l] + ca[t, l, j, k] + ca[ $\frac{e^{-tb_j} - 1}{b_j}, k, j, l$ ] +  $\delta aa \left[ \frac{1 - e^{-tb_j} - tb_j}{b_j^2}, j, k, j, l \right];$ 
Ad[a[t_, j_, k_]] [a[1, k_, l_]] /; DQ[j, k, l] & FreeQ[t, b_] :=
a[e^{tb_j}, k, l] + a[- $\frac{(-1 + e^{tb_j}) b_k}{b_j}, j, l$ ] + ca[ $\frac{-1 + e^{tb_j} (1 - tb_j)}{b_j}, k, k, l$ ] +
ca[ $\frac{b_j - e^{-tb_j} b_j + b_k + e^{tb_j} (-1 + tb_j) b_k}{b_j^2}, k, j, l$ ] +
ca[ $\frac{b_j + b_k + tb_j b_k - e^{tb_j} (b_j + b_k)}{b_j^2}, l, j, k$ ] +  $\delta aa \left[ \frac{1 + e^{tb_j} (-1 + tb_j)}{b_j^2}, j, k, k, l \right] +$ 
 $\delta aa \left[ \frac{1}{b_j^3} e^{-tb_j} (b_j + e^{2tb_j} (b_j + (2 - tb_j) b_k) - e^{tb_j} (2 b_k + b_j (2 + tb_k))) \right], j, k, j, l];$ 
Ad[a[t_, j_, k_]] [c[1, i_]] /; FreeQ[t, b_] & ({j, k} &cap; {i} == {}) := c[1, i];
Ad[a[t_, j_, k_]] [c[1, j_]] /; DQ[j, k] & FreeQ[t, b_] :=
c[1, j] + c[1 - e^{-tb_j}, k] +  $\delta a \left[ \frac{e^{-tb_j} - 1}{b_j}, j, k \right];$ 
Ad[a[t_, j_, k_]] [c[1, k_]] /; DQ[j, k] & FreeQ[t, b_] :=
c[e^{-tb_j}, k] +  $\delta a \left[ \frac{1 - e^{-tb_j}}{b_j}, j, k \right];$ 
Ad[x_β | x_c | x_δa | x_ca | x_δaa] [y_] := y + B[x, y];
Ad[x_] [a[f_, i_, j_]] /; f != 1 := Ad[x] [β[f]] ** Ad[x] [a[1, i, j]];
Ad[x_] [c[f_, i_]] /; f != 1 := Ad[x] [β[f]] ** Ad[x] [c[1, i]];
Ad[x_] [δa[f_, j_, k_]] := δ ** (β[f] ** Ad[x] [a[1, j, k]]);
Ad[x_] [ca[f_, i_, j_, k_]] := Ad[x] [c[f, i]] ** Ad[x] [a[1, j, k]];
Ad[x_] [δaa[f_, i_, j_, k_, l_]] := Ad[x] [δa[f, i, j]] ** Ad[x] [a[1, k, l]];
Ad[x_] [y_Plus] := Ad[x] /@ y;
Ad::ndef = "Ad[`1` is not defined on `2`.";
Ad[x_] [y_] := (Message[Ad::ndef, x, y]; Undefined);

```

AutoAd - Ad tests

```

Module[{t1, t2},
  {t1 = S[AutoAd[a[t, j, k]][#]],
   S[Ad[a[t, j, k]][#] - t1]}
] & @ a[1, i, k]
{a[e-t bj, i, k] + a[ $\frac{(1 - e^{-t b_j}) b_i}{b_j}$ , j, k] +
 ca[ $\frac{e^{-2 t b_j} b_i (-1 + e^{2 t b_j} - 2 e^{t b_j} t b_j)}{b_j^2}$ , k, j, k] + ca[ $\frac{e^{-2 t b_j} (1 - e^{t b_j} + e^{t b_j} t b_j)}{b_j}$ , k, i, k] +
 δa[ $\frac{e^{-2 t b_j} b_i (1 - e^{t b_j} + e^{t b_j} t b_j) \epsilon_2}{b_j^2}$ , j, k] + δa[ $-\frac{e^{-2 t b_j} (1 - e^{t b_j} + e^{t b_j} t b_j) \epsilon_2}{b_j}$ , i, k] +
 δaa[- $\frac{e^{-2 t b_j} b_i (-1 + e^{2 t b_j} - 2 e^{t b_j} t b_j)}{b_j^3}$ , j, k, j, k] +
 δaa[ $\frac{e^{-2 t b_j} (-1 + e^{t b_j} - e^{t b_j} t b_j)}{b_j^2}$ , i, k, j, k], 0}

AdTests[a[t, j, k]] = {β[f[bj, bk]], a[1, j, k], a[1, n, i], c[1, i],
  c[1, j], c[1, k], a[1, j, l], a[1, i, j], a[1, i, k], a[1, k, l]};

S[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ Take[AdTests[a[t, j, k]], All]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```

The semi group properties

```

Module[{t1, t2, t3, t4},
  t1 = Ad[a[t, j, k]][#] /.
    (h: (β | a | c | δa | ca | δaa)) [c_, r___] => h[SeriesCoefficient[c, {t, 0, 1}], r];
  t2 = B[a[1, j, k], #];
  t3 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
  t4 = # // Ad[a[t+s, j, k]];
  # -> S[{t1 == t2, t3 - t4}]
] & /@ AdTests[a[t, j, k]] // ColumnForm

β[f[bj, bk]] -> {True, 0}
a[1, j, k] -> {True, 0}
a[1, n, i] -> {True, 0}
c[1, i] -> {True, 0}
c[1, j] -> {True, 0}
c[1, k] -> {True, 0}
a[1, j, l] -> {True, 0}
a[1, i, j] -> {True, 0}
a[1, i, k] -> {True, 0}
a[1, k, l] -> {True, 0}

```

R

```

Switch[6,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t b_i, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[b_i, b_j]]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[b_j] + b_i f1[b_j]]],
  4, R[i_, j_][x_] :=
  x // Ad[a[1, i, j]] // Ad[c[f3[b_i, b_j], i]] // Ad[c[f4[b_i, b_j], j]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f5[b_i], i, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f6[b_i, b_j], j, i, j]],
  7, R[i_, j_][x_] :=
  x // Ad[a[1, i, j]] // Ad[ca[f5[b_i], i, i, j]] // Ad[ca[f6[b_i, b_j], j, i, j]]
];
VerifyR3[expr_] := Module[{lhs, rhs}, {
  lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // S;
  rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // S;
  S[lhs - rhs] == 0
}]

```

Verifying R3

```

VerifyR3 /@ {β[f[b1, b2, b3, b4]], c[f[b1, b2, b3, b4], 1], c[f[b1, b2, b3, b4], 2],
  c[f[b1, b2, b3, b4], 3], δa[f[b1, b2, b3, b4], 1, 2], δa[f[b1, b2, b3, b4], 1, 3],
  δa[f[b1, b2, b3, b4], 2, 3], a[f[b1, b2, b3, b4], 1, 4], a[f, 2, 4]}
{{True}, {True}, {True}, {True}, {True}, {True}, {True}, {True},
  {ca[2 e^{b1-b2} (-1 + e^{b2}) f f6[b1, b2] b1, 4, 2, 3] + ca[-\frac{e^{-b2} (-1 + e^{b2}) f (2 - 2 e^{b1} + (1 + e^{b1}) b1)}{b1},
    3, 2, 4] + ca[-2 e^{b1-b2} (-1 + e^{b2}) f f6[b1, b2] b2, 4, 1, 3] +
  ca[\frac{e^{-b2} (-1 + e^{b2}) f (2 - 2 e^{b1} + (1 + e^{b1}) b1) b2}{b1^2}, 3, 1, 4] +
  δaa[\frac{1}{b1^2} e^{-b2} (-1 + e^{b2}) f (2 - 2 e^{b1} + (1 + e^{b1}) b1 + 2 e^{b1} f6[b1, b2] b1^2), 1, 3, 2, 4] +
  δaa[-\frac{2 e^{b1-b2} (-1 + e^{b2}) f f6[b1, b2] b1}{b2}, 2, 3, 2, 4] +
  δaa[-\frac{e^{-b2} (-1 + e^{b2}) f (2 - 2 e^{b1} + (1 + e^{b1}) b1) b2}{b1^3}, 1, 3, 1, 4] == 0}}

```