Pensieve header: Verification of the backie commutation relation, $[a_{jk}, a_{kj}]$.

The bracket

On the elements β, a, c, δa, ca, δaa.

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];

Simp[expr_] := Simplify[expr];
S[β[f_]] := β[Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[δa[f_, i_, j_]] := δa[Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[δaa[f_, i_, j_, k_, l_]] := δaa[Simp[f], i, j, k, l];
S[expr_] := expr /. (λ_β | λ_a | λ_δa | λ_c | λ_ca | λ_δaa) :> S[λ];

β[0] := 0;
β /: β[f_] + β[g_] := β[f + g] // S;
β /: g_ * β[f_] := β[g f] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f + g, j, k] // S;
a /: g_ * a[f_, j_, k_] := a[g f, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f + g, j] // S;
c /: g_ * c[f_, j_] := c[g f, j] // S;
δa[0, _, _] := 0;
δa /: δa[f_, j_, k_] + δa[g_, j_, k_] := δa[f + g, j, k] // S;
δa /: g_ * δa[f_, j_, k_] := δa[g f, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f + g, j, k, l] // S;
ca /: g_ * ca[f_, j_, k_, l_] := ca[g f, j, k, l] // S;
δaa[0, _, _, _, _] := 0;
δaa /: δaa[f_, i_, j_, k_, l_] + δaa[g_, i_, j_, k_, l_] :=
  δaa[f + g, i, j, k, l] // S;
δaa /: g_ * δaa[f_, i_, j_, k_, l_] := δaa[g f, i, j, k, l] // S;
```

NonCommutativeMultiply

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_, 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;

β[f_] ** a[g_, j_, k_] := a[f g, j, k];
β[f_] ** c[g_, j_] := c[f g, j];
c[g_, j_] ** β[f_] := c[f g, j];
β[f_] ** δa[g_, j_, k_] := δa[f g, j, k];
β[f_] ** ca[g_, i_, j_, k_] := ca[f g, i, j, k];
ca[g_, i_, j_, k_] ** β[f_] := ca[f g, i, j, k];
β[f_] ** δaa[g_, i_, j_, k_, l_] := δaa[f g, i, j, k, l];
δa[g_, j_, k_] ** β[f_] := δa[f g, j, k];
δ ** a[f_, i_, j_] := δa[f, i, j];
c[f_, i_] ** a[g_, j_, k_] := ca[f g, i, j, k];
a[f_, i_, j_] ** δa[g_, k_, l_] := δaa[f g, i, j, k, l];
δa[f_, i_, j_] ** a[g_, k_, l_] := δaa[f g, i, j, k, l];

δ ** _c = 0;
δ ** _δa = 0;
δ ** _ca = 0;
δ ** _δaa = 0;
_c ** _c = 0;
_c ** _δa = _δa ** _c = 0;
_c ** _ca = _ca ** _c = 0;
_c ** _δaa = _δaa ** _c = 0;
_δa ** _δa = 0;
_δa ** _δaa = _δaa ** _δa = 0;
_δa ** _ca = _ca ** _δa = 0;

NonCommutativeMultiply::ndef =
 "NonCommuatativeMultiply is not defined on {`1`,`2`}."
NonCommutativeMultiply[x_, y_] :=
  (Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
```

```
NonCommuatativeMultiply is not defined on {`1`,`2`}.
```

Bracket Generalities

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

The γ shortcuts

```
γ[f_, j_, k_] := δa[f, j, k] - c[b_j f, k] // S;
γ[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;
ac[f_, j_, k_, l_] := ca[f, l, j, k] + B[a[1, j, k], c[f, l]];
adown[f_, j_] := a[f, j, j] + β[ϵ1 f b_j] + c[ϵ1 f, j];
```

Fundamental Brackets

a-β, a-c, a-a, AS

```
B[a[j_, k_], β[g_]] := γ[∂_{b_j} g - ∂_{b_k} g, j, k];
B[β[g_], a[j_, k_]] := -B[a[j, k], β[g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k} ⋂ {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // S;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] - γ[1, j, k, l] // S;
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
(* backie *) B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[ϵ5 b_j, k, j] - a[ϵ5 b_k, j, k] + a[ϵ5 b_j, k, k] - a[ϵ5 b_k, j, j] + ca[ϵ4, k, k, j] -
   ca[ϵ4, j, j, k] + ca[ϵ4, k, j, j] - ca[ϵ4, j, k, k] + γ[ϵ4, j, k] - γ[ϵ4, k, j];
(* [tail, selfie] *) B[a[j_, k_], a[j_, j_]] /; DQ[j, k] := γ[-ϵ6, j, k] // S;
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
(* [head, selfie] *) B[a[j_, k_], a[k_, k_]] /; DQ[j, k] := γ[-ϵ7, j, k] // S;
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[k, k]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] := γ[-f g, j, k];
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] := γ[f g, j, k];
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k} ⋂ {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];
```

Vanishing brackets

```
B[_β, _β | δ | _c | _δa | _ca | _δaa] = 0;
B[_β | δ | _c | _δa | _ca | _δaa, _β] = 0;
B[δ | _c | _δa | _ca | _δaa, δ | _c | _δa | _ca | _δaa] = 0;
```

Composite Brackets

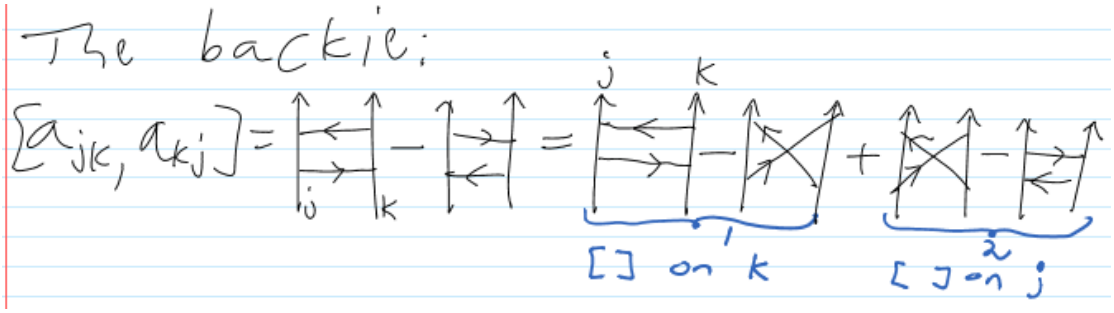
```
B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[1, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[1, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[1, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[1, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];
```

```
B::ndef = "B is not defined on {`1`,`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
```

```
B is not defined on {`1`,`2`}.
```

Backie Verification

From pensieve://2015-05/Selfie and backie commutation relations:




```
B[a[j, k], a[k, jj]]
```

$$a[b_j, k, jj] + a[-b_k, j, jj] + ca[-1, jj, j, k] + ca[1, k, j, jj]$$

```
T1 = a[b_j, k, j] + a[-b_k, j, j] + ac[-1, j, k, j] + ca[1, k, j, j]
```

$$a[b_j, k, j] + a[-b_k, j, j] + c[-b_j, k] + ca[-1, j, j, k] + ca[1, k, j, j] + \delta a[1, j, k]$$

```
B[a[j, kk], a[k, j]]
```

$$a[b_j, k, kk] + a[-b_k, j, kk] + ca[-1, j, k, kk] + ca[1, kk, k, j]$$

```
T2 = a[b_j, k, k] + a[-b_k, j, k] + ca[-1, j, k, k] + ac[1, k, j, k]
```

```
a[b_j, k, k] + a[-b_k, j, k] + c[b_k, j] + ca[-1, j, k, k] + ca[1, k, k, j] + δa[-1, k, j]
```

```
B[a[j, k], a[k, j]] - T1 - T2 /. {ϵ4|5 → 1}
```

```
0
```