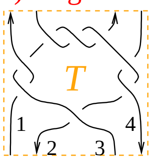
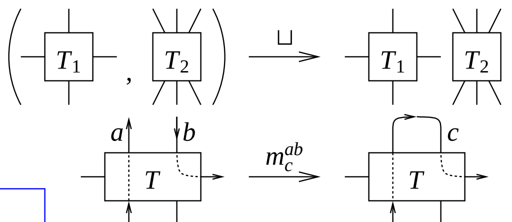
Polynomial Time Knot Polynomials, A

Abstrant. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



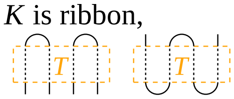


(meta-associativity: $m_a^{ab} /\!/ m_a^{ac} = m_b^{bc} /\!/ m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,

$z(K) \in \{cl_2(\zeta) \colon cl_1(\zeta) = 1\}$.



cl_1: trivial cl_2: ribbon

$U \in \mathcal{T}_n$, cl_1, $\mathcal{T}_{2n}$, cl_2, $K \in \mathcal{T}_1$

(Genus and crossing number are also definable properties).

A blackboard aside on genus?

Faster is better, leaner is meaner!

Theorem 1.

$\exists!$ an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\ \sqcup\ } \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\substack{T_a, T_b \to T_c\\ \mu := 1-\beta}]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a;\ {}_a\!\nearrow\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\nearrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm 1}\\ b & 0 & T_a^{\pm 1}\end{array}\right)$.

In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
  M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a} λ  ∂_{t_a,h_b} λ  ∂_{t_a} λ;
                          ∂_{t_b,h_a} λ  ∂_{t_b,h_b} λ  ∂_{t_b} λ;
                          ∂_{h_a} λ      ∂_{h_b} λ      λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ;
                             ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;
```

Meta-Associativity

Runs.

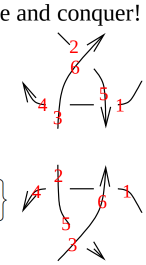
```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1;
                           α21 α22 α23 θ2;
                           α31 α32 α33 θ3;
                           ϕ1  ϕ2  ϕ3  Ξ).{h1, h2, h3, hS}];
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
```

```
True
```

R3 … divide and conquer!

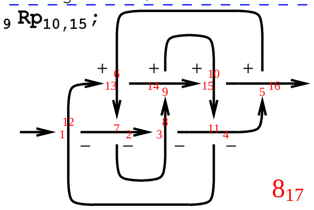
```
{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
 Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$




```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$

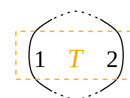


8_{17}

Closed Components.

The Halacheva trace tr_c satisfies $m_c^{ab} /\!/ tr_c = m_c^{ba} /\!/ tr_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

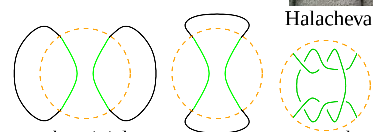
$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{tr_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi + \psi\theta/\mu\end{array}$$



Halacheva

```
tr_{c_}[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c} λ  ∂_{t_c} λ; ∂_{h_c} λ  λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m_{12→1} // tr_1) == (ζ // m_{21→1} // tr_1)
```

```
True
```



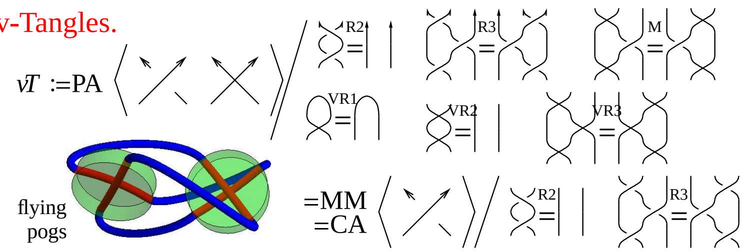
cl_1: trivial cl_2: ribbon example

Weaknesses.

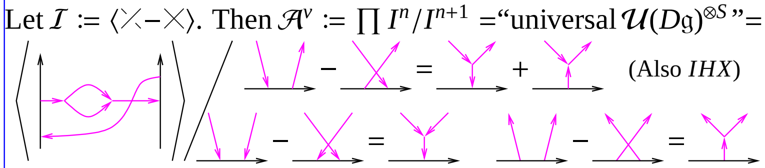
- m_c^{ab} and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



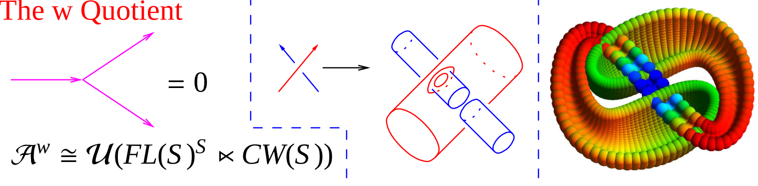
Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem.

[EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z \colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

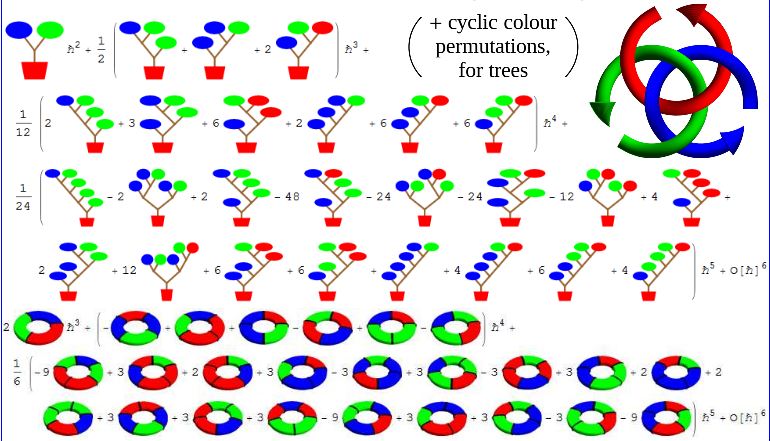
The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

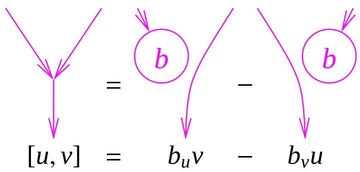
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:

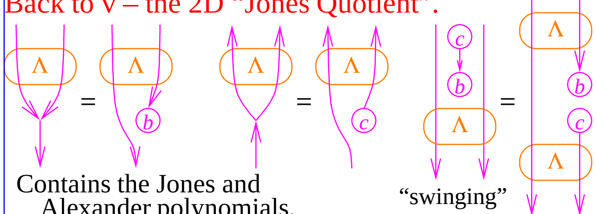


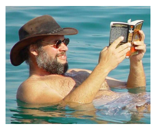
(I have a fancy free-Lie calculator!) (ωεβ/FLD) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



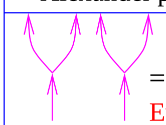
Back to v – the 2D "Jones Quotient".





V. Jones

Contains the Jones and Alexander polynomials, . . . yet still too hard!



The OneCo Quotient. Likely related to [ADO]

= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

References.

[ADO] Y. Akutsu, T. Deguchi, and T. Ohtsuki, *Invariants of Colored Links*, J. of Knot Theory and its Ramifications **1-2** (1992) 161–184.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Help Needed!

I'm slow and feeble-minded.

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a,b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a,b\}$, such that

The Abstract Context

$$\text{meta-associativity:}\quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

$$\text{meta-locality:}\quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with

$$\Gamma_1(S) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := R_S\langle S\rangle).$$

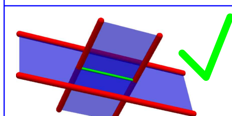
Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

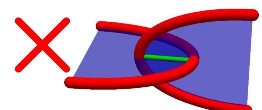
$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c\right)$$

What's done? The braid part, with still-ugly formulas.

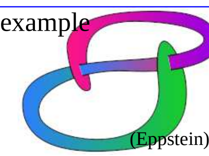
What's missing? A lot of concept- and detail-sensitive work towards m^{ab}_{1c}, α^{ab}, and ρ^{ab}_c. The "ribbon element".



a ribbon singularity



a clasp singularity



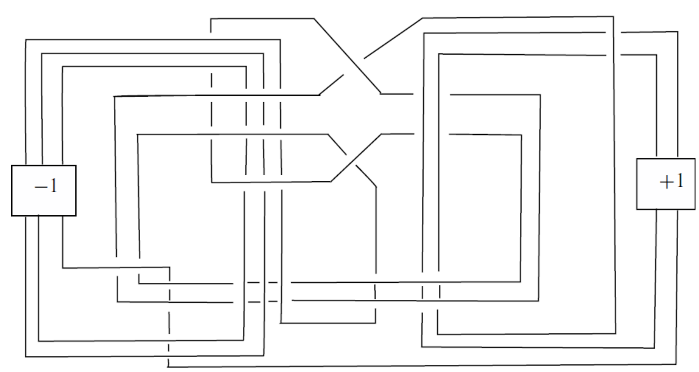
example (Eppstein)

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

[GST]: a slice knot that might not be ribbon (48 crossings).



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

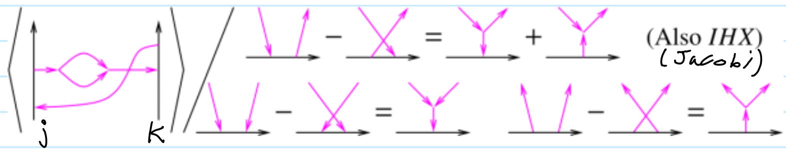
www.katlas.org

PolyPoly Extras

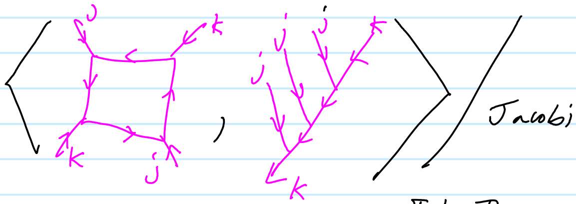
Monday, August 24, 2015 3:10 AM

Dror Bar-Natan, Les Diablerets, August 2015
http://drorbn.net/LD

$\mathcal{A}^v =$



$P\mathcal{A}^v/(\ldots = 0) =$



$\| \quad P\mathcal{A}^w = P\mathcal{A}^v/1co$

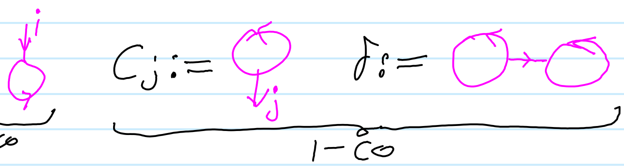
So

$P\mathcal{A}^w(\uparrow_S)/(\ldots = \ldots - \ldots) = \hat{R}_S \oplus M_{S\times S}(\hat{R}_S)$ // $\oplus[b_i]$

and the rest is (hard!) calculations, which lead to a simple rational-function result.

$P\mathcal{A}^v/(\ldots = 0) =$

So with $b_i :=$ …, $c_j :=$ …, $\delta :=$ …,



$(P\mathcal{A}^v/2co)/_{2D} \subset$

$\hat{R}_S \oplus M_{S\times S}(\hat{R}_S) \oplus \hat{R}_S \ldots \oplus \delta \hat{R}_S \ldots \oplus \hat{R}_S \ldots \oplus \delta \hat{R}_S \ldots$

$= V_S + V_S^{\otimes 2} + V_S + V_S^{\otimes 2} + V_S^{\otimes 3} + (S^2(V_S))^{\otimes 2}$

[The product law is awful, but experience shows that things simplify....]

Stitching is clearly possible, but I still don't have explicit formulas.

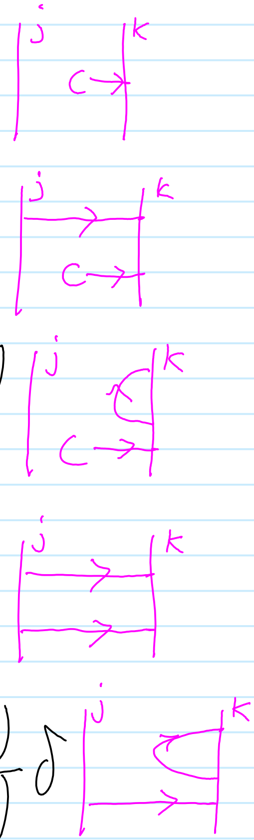
Proposition The element R_{ij} given below solves the YB equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathcal{A}^v/2co/2D$!

$R_{jk} = e^{j\text{—}k} e^{\rho}$, with

$$\rho = -\phi_2(b_j)\,[\ldots] + \frac{\phi_2(b_j)}{b_j}\,[\ldots] + \frac{\phi_1(b_j)\phi_2(b_k)}{b_k\phi_1(b_k)}\,[\ldots] - \frac{\phi_2(b_j)}{b_j^2}\delta\,[\ldots] - \frac{\phi_1(b_j)\phi_2(b_k)}{b_j b_k \phi_1(b_k)}\delta\,[\ldots]$$



where $\phi_1(x) = e^{-x} - 1$

and $\phi_2(x) = \dfrac{(x+2)e^{-x} - 2 + x}{2x}$

Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages FreeLie` and AwCalculus` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌢, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y},\ \frac{1}{24}\overline{x\overline{\overline{xy}y}},\ -\frac{1}{720}\overline{x\overline{x\overline{x\overline{xy}}}}+\frac{1}{180}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{180}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{120}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},\ -\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1440}+\frac{1}{360}\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}+\frac{1}{240}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{1}{720}\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}-\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{1440},\ \ldots\Big]$$

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad-1), x][u] + adSeries[ad/(e^ad-1), y][u] -
       adSeries[ad/(e^ad-1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}}+\frac{1}{80}\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{5040}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1344}+\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{15120}+\frac{1}{840}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{3360}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{6720}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1260}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1680}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{10080},\ \ldots\Big],$$

$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}}+\frac{1}{120}\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{10080}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{2016}+\frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1890}+\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{1120}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{5040}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{2520}+\frac{1}{840}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1260}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{5040},\ \ldots\Big]\Big\}$$

Meaningless calculations.

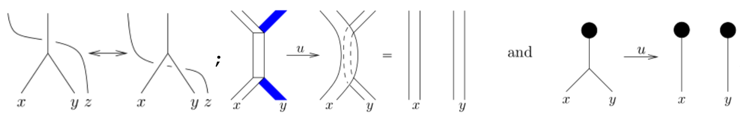
```
{b[F, G], tr_x[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{48}\overline{\overline{\overline{xy}y}y},\ \frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}-\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}-\frac{\overline{\overline{xy}\overline{\overline{xy}y}}}{1440}-\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{360}\overline{\overline{\overline{\overline{xy}y}y}y},\ \frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1440}-\frac{1}{480}\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}-\frac{1}{288}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}-\frac{7\,\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{2880}+\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{2880},\ \ldots\Big],$$

$$\mathrm{CWS}\Big[-\frac{\overline{y}}{6},\ \frac{\overline{yy}}{24},\ \frac{\overline{xxy}}{180}+\frac{\overline{xyy}}{80}-\frac{\overline{yyy}}{360},\ -\frac{\overline{xxyy}}{180}+\frac{\overline{xyxy}}{240}-\frac{\overline{xyyy}}{240}-\frac{\overline{yyyy}}{1440},\ -\frac{\overline{xxxxy}}{5040}+\frac{\overline{xxxyy}}{6720}-\frac{\overline{xxyxy}}{1120}+\frac{2\,\overline{xxyyy}}{945}-\frac{\overline{xyxyy}}{336}+\frac{\overline{xyyyy}}{6720}+\frac{\overline{yyyyy}}{10080},\ \frac{\overline{xxxxyy}}{3360}-\frac{\overline{xxxyxy}}{1344}-\frac{\overline{xxxyyy}}{2240}+\frac{\overline{xxyxxy}}{2016}+\frac{13\,\overline{xxyxyy}}{10080}+\frac{\overline{xxyyxy}}{1680}-\frac{\overline{xxyyyy}}{3780}-\frac{\overline{xyxyxy}}{840}+\frac{\overline{xyxyyy}}{5040}+\frac{\overline{xyyxyy}}{2240}+\frac{\overline{xyyyyy}}{6720}+\frac{\overline{yyyyyy}}{60480},\ \ldots\Big]\Big\}$$

(Also implemented: ∂_λ and derivations in general, tb, e^{∂_λ} and morphisms in general, div, j, Drinfel'd-Kohno, etc.)

The [BND] "vertex" equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop :
Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

$$\Big\{\mathrm{Es}\Big[\Big\langle \overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\overline{x\overline{xy}}}}{5760}-\frac{7\,\overline{x\overline{\overline{xy}y}}}{5760}+\frac{\overline{\overline{\overline{xy}y}y}}{1440},\ 0,\ -\frac{31\,\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{967680}+\frac{31\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{83\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{967680}-\frac{31\,\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{725760}-\frac{31\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{645120}+\frac{13\,\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{241920}+\frac{101\,\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1451520}+\frac{527\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{5806080}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{60480},\ \ldots\Big],$$

$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\overline{x\overline{xy}}}}{5760}-\frac{1}{720}\overline{x\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{\overline{xy}y}y},\ -\frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{7680}+\frac{\overline{x\overline{x\overline{\overline{xy}y}}}}{3840}-\frac{\overline{\overline{x\overline{xy}}\overline{xy}}}{6912},\ -\frac{\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{645120}+\frac{23\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{161280}-\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{22680}-\frac{41\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{580608}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{15120}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{12096}+\frac{71\,\overline{x\overline{\overline{\overline{xy}y}}\overline{xy}}}{483840}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{30240},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880}+\frac{\overline{xxyy}}{2880}+\frac{\overline{xyxy}}{5760}+\frac{\overline{xyyy}}{2880},\ 0,\ -\frac{\overline{xxxxxy}}{120960}-\frac{\overline{xxxxyy}}{120960}-\frac{\overline{xxxyxy}}{120960}-\frac{\overline{xxxyyy}}{120960}-\frac{\overline{xxyxxy}}{241920}-\frac{\overline{xxyxyy}}{120960}-\frac{\overline{xxyyxy}}{120960}-\frac{\overline{xxyyyy}}{120960}-\frac{\overline{xyxyxy}}{362880}-\frac{\overline{xyxyyy}}{120960}-\frac{\overline{xyyxyy}}{241920}-\frac{\overline{xyyyyy}}{120960},\ \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11520},\ 0,\ -\frac{\overline{xxxxxx}}{725760},\ \ldots\Big]\Big\}$$

From V to F to KV following [AT].

```
logF = Λ[V]⟦1⟧ // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[(e^ad-1)/ad, logF, tb]
```

⟨x → LS[y/2, xy/6, 1/24 xyy, −1/180 xxxy + 1/80 xxyy + 1/360 xyyy,
−1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy −
xyyyy/1440, xxxxxy/5040 − xxxxyy/1344 + 13 xxxyyy/15120 + 1/840 xxyxyy +
xxxyxy/3360 + xxyyyy/6720 + xyxyyy/1260 + xxyyxy/1680 − xyyyyy/10080, …],
y → LS[0, xy/12, 1/24 xyy, −1/360 xxxy + 1/120 xxyy + 1/180 xyyy,
−1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy −
xyyyy/1440, xxxxxy/10080 − xxxxyy/2016 + xxxyyy/1890 + xxyxyy/1120 + xxxyxy/5040 +
xxyyyy/2520 + 1/840 xyxyyy + xxyyxy/1260 − xyyyyy/5040, …]⟩

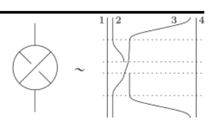
Solving for an associator Φ.

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ −Φ) ∧
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

DKS[0, 1/24 t13 t23, 0, −7 t13 t23 t23 t23/5760 + 7 t13 t13 t23 t23/5760 − t13 t13 t13 t23/1440,
0, 31 t13 t23 t23 t23 t23 t23/967680 − 157 t13 t13 t23 t23 t13 t23/1935360 −
31 t13 t23 t13 t23 t23 t23/387072 − 31 t13 t13 t23 t23 t23 t23/483840 +
11 t13 t13 t13 t13 t23 t13 t23/290304 + 31 t13 t13 t23 t13 t23 t23/725760 + 83 t13 t13 t13 t23 t23 t23/967680 −
13 t13 t13 t13 t13 t23 t23/241920 + t13 t13 t13 t13 t13 t23/60480, …]

The "buckle" Z_B, from Φ.



```
R = DKS[t[1, 2] / 2];
ZB = (−Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (−Φ)^σ[1,2,3] **
   Φ^σ[12,3,4];
ZB@{4}
```

DKS[t23/2, −1/12 t13 t23 − 1/24 t14 t24 + 1/24 t14 t34 + 1/12 t24 t34,
0, t13 t23 t23 t23/5760 + 7 t14 t24 t24 t24/5760 + t14 t34 t24 t24/1920 −
t14 t34 t34 t24/1920 − 7 t14 t34 t34 t34/5760 − t24 t34 t34 t34/5760 + t14 t24 t34 t24/1920 +
t14 t24 t14 t34/1920 − t14 t34 t24 t34/1920 − 1/720 t13 t13 t23 t23 +
1/720 t13 t13 t13 t23 − 7 t14 t14 t24 t24/5760 + 7 t14 t14 t34 t34/5760 −
t14 t24 t34 t34/5760 + t14 t14 t14 t24/1440 − t14 t14 t14 t34/1440 − 1/960 t14 t14 t24 t34 +
t14 t24 t24 t34/5760 − 1/960 t24 t24 t34 t34 − t24 t24 t24 t34/5760, …]

V from Z_B, following [AET, BND].

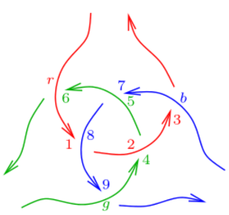
```
(El[ZB // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη¹ // tη³ //
    hη² // hη⁴ // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])⟦
  1⟧
```

⟨1 → LS[0, −12/24, 0, 71112/5760 − 71122/5760 + 1222/1440, 0,
−3111112/967680 + 3111122/483840 − 8311222/967680 − 3111222/725760 − 3111212/645120 +
13112222/241920 + 10112122 2/1451520 + 52711221 2/5806080 − 122222/60480, …],
2 → LS[1/2, −12/12, 0, 1112/5760 − 1/720 1122 + 1/720 1222,
−11112/7680 + 11122/3840 − 11212/6912,
−111112/645120 + 2311122/483840 − 1311222 2/161280 − 112122/22680 − 4111212/580608 +
112222/15120 + 121222/12096 + 7111221 2/483840 − 122222/30240, …]⟩

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```

```
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ⟦1⟧r@{5}, ξ⟦2⟧@{5}} // Print
```



;

{LS[0, bg, 1/2 bbg + bgr + 1/2 bgg,
1/6 bbbg + 1/2 bbgr + 1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg,
1/24 bbbbg + 1/6 bbbgr + 1/4 bbggr + 1/12 bbbgg +
1/4 bbgrr + 1/6 bgggr + 1/4 bggrr − bbgrg +
1/12 bbggg − 2 bbrgg + 1/6 bgrrr + 1/2 bgbgr −
bgbrg − 1/12 bbgbg − 1/2 bgrgr + 1/24 bgggg, …],
CWS[0, 0, 2 bgr, bbgr − bgbr + bggr − bgrg + bgrr − brgr, bbbgr/3 −
bbgbr/2 + bbggr/2 + bbgrg/2 + bbgrr/2 + bbrbg/2 − 3 bbrgr/2 + bgbrr/2 − 3 bggbr/2 + bggqr/3 −
bggrg/2 + bggrr/2 + bgrgg/2 − 3 bgrrg/2 + bgrrr/3 + brggr/2 − brgrr/2 + brrgr/2, …]}

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv:not.yet×2.

Warning. Fidgety!