Videos of all talks are available at their respective web pages

Dror Bar-Natan: Talks: LesDiablerets-1508:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

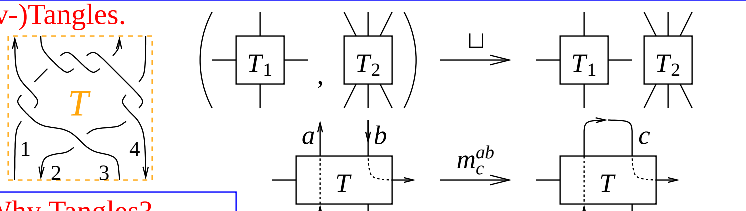
Work in Progress on

Polynomial Time Knot Polynomials, A

Abstrant. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.

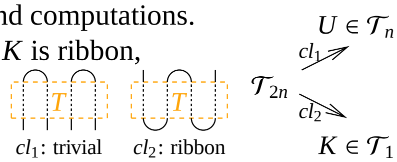


(meta-associativity: $m_a^{ab}/\!\!/m_a^{ac} = m_b^{bc}/\!\!/m_a^{ab}$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta)\colon cl_1(\zeta)=1\}$.

(Genus and crossing number are also definable properties).



A blackboard aside on genus?

Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array},\ \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\ \sqcup\ } \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c,\ \mu:=1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a;\ {}_a\nearrow_b,\ {}_b\nwarrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det{}'(A-I)/(1-T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
  M = Outer[Factor[∂_{h_#2 t_#1} λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, h_# & /@ S, ω]];
  M // MatrixForm];

Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ;
                           ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ;
                           ∂_{h_a}λ ∂_{h_b}λ λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + αδ/μ  ϵ + δθ/μ; ϕ + αψ/μ  Ξ + ψθ/μ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1-T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1/T_a;
```

Meta-Associativity

Runs.

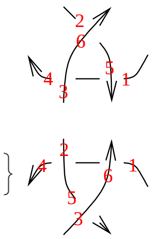
```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```

True

R3 … divide and conquer!

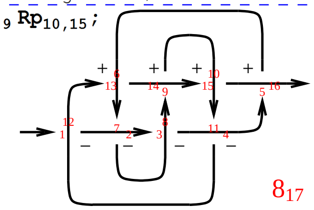
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



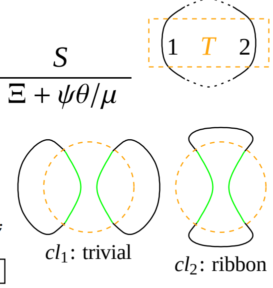
Closed Components. The Halacheva trace tr_c satisfies $m_c^{ab}/\!\!/tr_c = m_c^{ba}/\!\!/tr_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{tr_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi+\psi\theta/\mu\end{array}$$

```
tr_c_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c}λ ∂_{t_c}λ; ∂_{h_c}λ λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

True

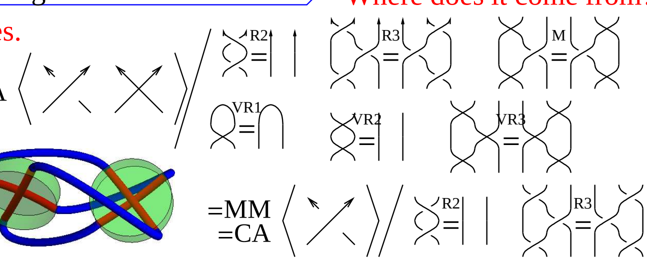
Halacheva



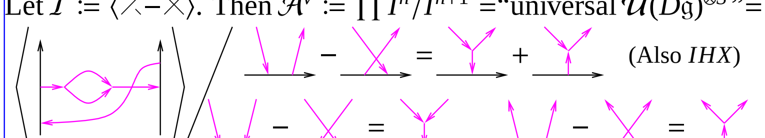
Weaknesses. • m_c^{ab} and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=

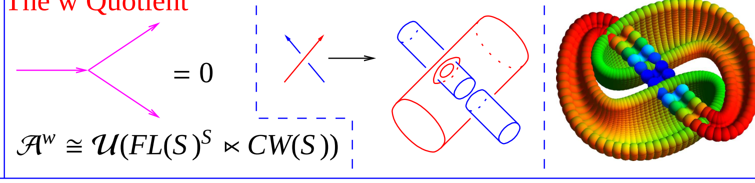


Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

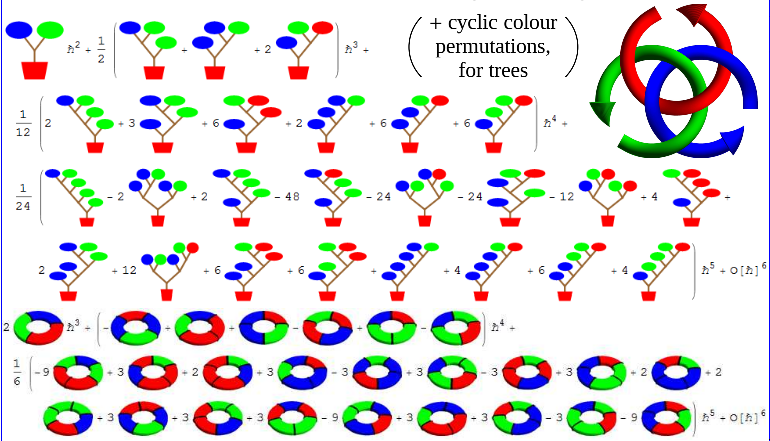
The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

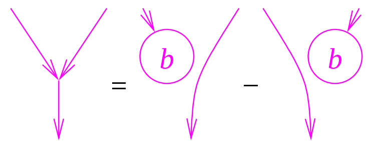
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:

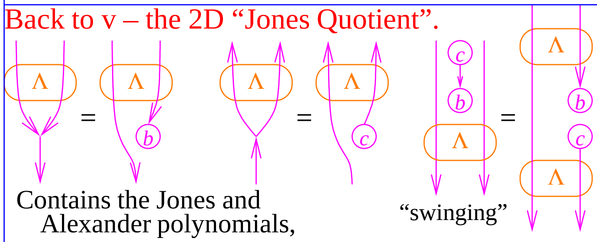


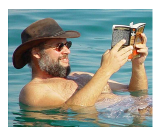
(I have a fancy free-Lie calculator!) (ωεβ/FLD) **Nice, but too hard!**

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



Back to v – the 2D "Jones Quotient".

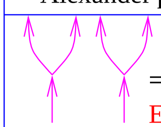




V. Jones

Contains the Jones and Alexander polynomials, …yet still too hard!

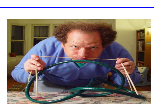
The OneCo Quotient. Likely related to [ADO]



= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

References.
[ADO] Y. Akutsu, T. Deguchi, and T. Ohtsuki, *Invariants of Colored Links*, J. of Knot Theory and its Ramifications **1-2** (1992) 161–184.
[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.
[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.
[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.
[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.
[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.
[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.
[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.
[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.



Help Needed!

I'm slow and feeble-minded.

The Abstract Context

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*$: $M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and m^{ab}_c: $M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

$$\text{meta-associativity:}\quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

$$\text{meta-locality:}\quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 \colon PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} \colon PvT \to \Gamma_{01}$, with

$$\Gamma_1(S) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := R_S\langle S\rangle).$$

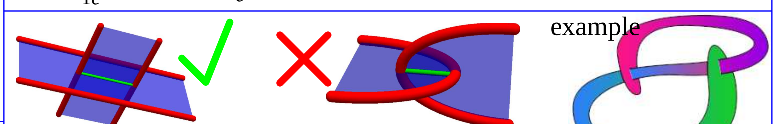
Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear m^{ab}_{1c}: $\Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action α^{ab}: $\Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) ρ^{ab}_c: $\Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c\right)$$

What's done? The braid part, with still-ugly formulas.

What's missing? A lot of concept- and detail-sensitive work towards m^{ab}_{1c}, α^{ab}, and ρ^{ab}_c. The "ribbon element".



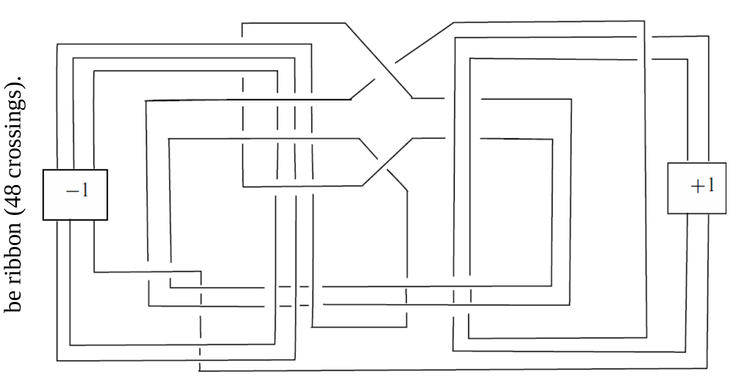
a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

[GST]: a slice knot that might not be ribbon (48 crossings).

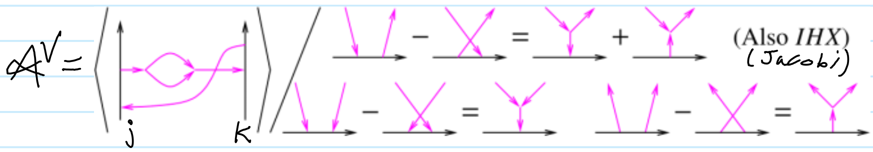


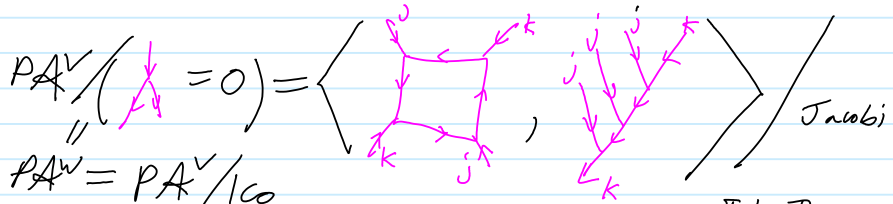
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

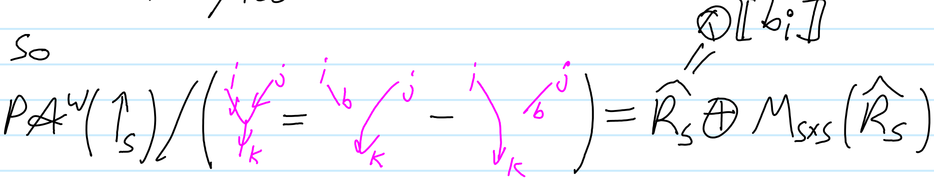
Dror Bar-Natan, Les Diablerets, August 2015
http://drorbn.net/LD



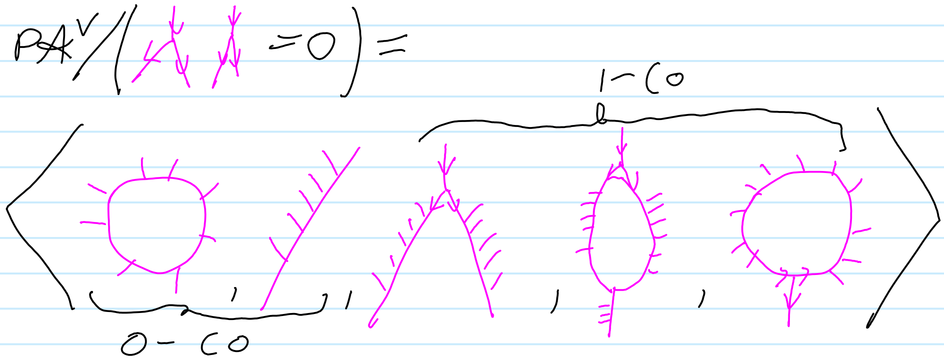


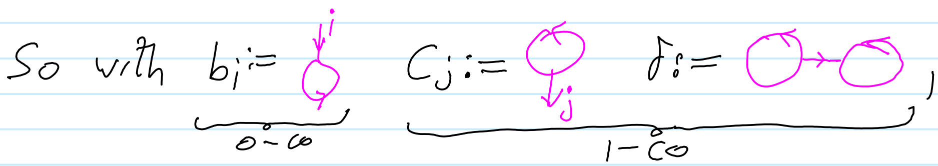
$P\mathcal{A}^w = P\mathcal{A}^v/1co$

So

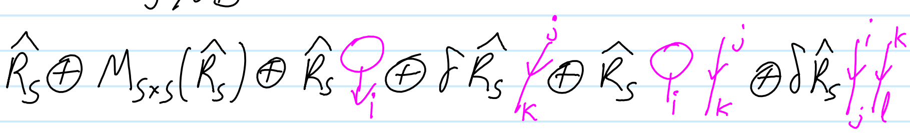


and the rest is (hard!) calculations, which lead to a simple rational-function result.





$(P\mathcal{A}^v/2co)/_{2D} \subset$



$$= V_s + V_s^{\otimes 2} + V_s + V_s^{\otimes 2} + V_s^{\otimes 3} + (S^2(V_s))^{\otimes 2}$$

[The product law is awful, but experience shows that things simplify....]

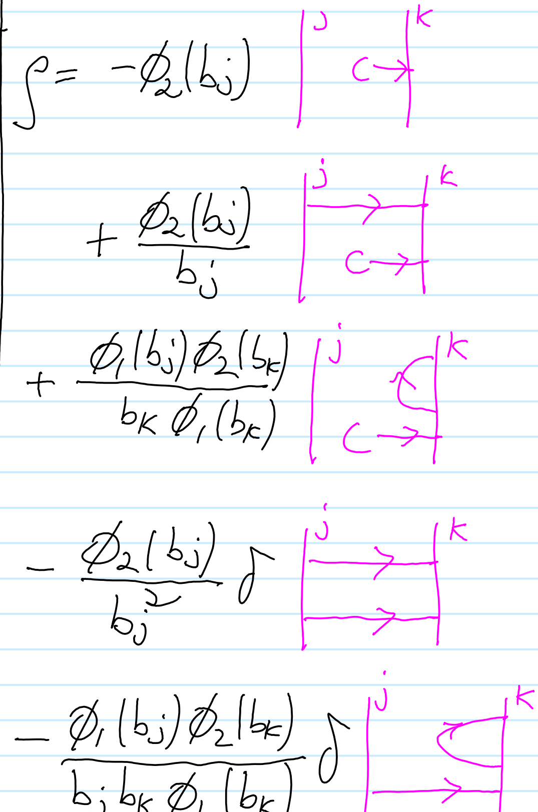
Stitching is clearly possible, but I still don't have explicit formulas.

Proposition The element R_{ij} given below solves the YB equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $\mathcal{A}^v/2co/2D$!

$R_{jk} = e^{j\longrightarrow k} e^{\varsigma}$, with



where $\phi_1(x) = e^{-x} - 1$

and $\phi_2(x) = \dfrac{(x+2)e^{-x} - 2 + x}{2x}$

Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages FreeLie` and AwCalculus` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌒, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

LS[x + y, xy/2, 1/12 xxy + 1/12 xyy, 1/24 xxyy,
 −1/720 xxxxy + 1/180 xxxyy + 1/180 xxyyy + 1/120 xyxyy + 1/360 xxyxy − 1/720 xyyyy, −xxxxyy/1440 + 1/360 xxxyyy + 1/240 xxyxyy + 1/720 xxxyxy − xxyyyy/1440, …]

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad-1), x][u] + adSeries[ad/(e^ad-1), y][u] -
      adSeries[ad/(e^ad-1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

{LS[y/2, xy/6, 1/24 xyy, −1/180 xxxy + 1/80 xxyy + 1/360 xyyy,
 −1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy − xyyyy/1440, xxxxxy/5040 − xxxxyy/1344 + 13 xxxyyy/15120 + 1/840 xxyxyy + xxxyxy/3360 + xxyyyy/6720 + xyxyyy/1260 + xxyyxy/1680 − xyyyyy/10080, …],
LS[0, xy/12, 1/24 xyy, −1/360 xxxy + 1/120 xxyy + 1/180 xyyy,
 −1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy − xyyyy/1440, xxxxxy/10080 − xxxxyy/2016 + xxxyyy/1890 + xxyxyy/1120 + xxxyxy/5040 + xxyyyy/2520 + 1/840 xyxyyy + xxyyxy/1260 − xyyyyy/5040, …]}

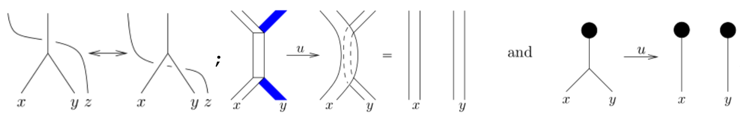
Meaningless calculations.

```
{b[F, G], tr_x[F]}
```

{LS[0, 0, −1/24 xyy, −1/48 xyyy, 1/720 xxxyy − 1/240 xxyyy − xyxyy/1440 − 1/720 xxyxy − 1/360 xyyyy, xxxyyy/1440 − 1/480 xxyyyy − 1/288 xyxyyy − 7 xxyyxy/2880 + xyyyyy/2880, …],
CWS[−y/6, yy/24, xxy/180 + xyy/80 − yyy/360, −xxyy/180 + xyxy/240 − xyyy/240 − yyyy/1440,
 −xxxxy/5040 + xxxyy/6720 − xxyxy/1120 + 2 xxyyy/945 − xyxyy/336 + xyyyy/6720 + yyyyy/10080,
 xxxxyy/3360 − xxxyxy/1344 − xxxyyy/2240 + xxyxxy/2016 + 13 xxyxyy/10080 + xxyyxy/1680 − xxyyyy/3780 − xyxyxy/840 + xyxyyy/5040 + xyyxyy/2240 + xyyyyy/6720 + yyyyyy/60480, …]}

(Also implemented: ∂_λ and derivations in general, tb, e^{∂_λ} and morphisms in general, div, j, Drinfel'd-Kohno, etc.)

The [BND] "vertex" equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

{Es[⟨x → LS[0, −xy/24, 0, 7 xxxy/5760 − 7 xxyy/5760 + xyyy/1440, 0,
 −31 xxxxxy/967680 + 31 xxxxyy/483840 − 83 xxxyyy/967680 − 31 xxyxyy/725760 − 31 xxxyxy/645120 + 13 xxyyyy/241920 + 101 xyxyyy/1451520 + 527 xxyyxy/5806080 − xyyyyy/60480, …],
 y → LS[x/2, −xy/12, 0, xxxy/5760 − 1/720 xxyy + 1/720 xyyy, −xxxxy/7680 + xxxyy/3840 − xxyxy/6912, −xxxxxy/645120 + 23 xxxxyy/483840 − 13 xxxyyy/161280 − xxyxyy/22680 − 41 xxxyxy/580608 + xxyyyy/15120 + xyxyyy/12096 + 71 xxyyxy/483840 − xyyyyy/30240, …]⟩,
CWS[0, −xy/48, 0, xxxy/2880 + xxyy/2880 + xyxy/5760 + xyyy/2880, 0,
 −xxxxxy/120960 − xxxxyy/120960 − xxxyxy/120960 − xxxyyy/120960 − xxyxxy/241920 − xxyxyy/120960 − xxyyxy/120960 − xxyyyy/120960 − xyxyxy/362880 − xyxyyy/120960 − xyyxyy/241920 − xyyyyy/120960, …]],
CWS[0, −xx/96, 0, −xxxx/11520, 0, −xxxxxx/725760, …]}

From V to F to KV following [AT].

```
logF = Λ[V]⟦1⟧ // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[e^ad-1/ad, logF, tb]
```

⟨x → LS[y/2, xy/6, 1/24 xyy, −1/180 x xxy + 1/80 x xyy + 1/360 xyyy,
−1/720 xx xyy + 1/240 x xyyy + 1/240 xy xyy + 1/720 xxy xy − xyyyy/1440, xxxxxy/5040 − xxxxyy/1344 + 13 xxxyyy/15120 + 1/840 x xy xyy + x xxyxy/3360 + x xyyyy/6720 + xy xyyy/1260 + x xyyxy/1680 − xyyyyy/10080, …],
y → LS[0, xy/12, 1/24 xyy, −1/360 x xxy + 1/120 x xyy + 1/180 xyyy,
−1/720 xx xyy + 1/240 x xyyy + 1/240 xy xyy + 1/720 xxy xy − xyyyy/1440, xxxxxy/10080 − xxxxyy/2016 + xxxyyy/1890 + xxyxyy/1120 + x xxyxy/5040 + x xyyyy/2520 + 1/840 xy xyyy + x xyyxy/1260 − xyyyyy/5040, …]⟩

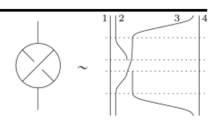
Solving for an associator Φ.

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ -Φ) ∧
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

DKS[0, 1/24 t₁₃t₂₃, 0, −7 t₁₃t₂₃t₂₃t₂₃/5760 + 7 t₁₃t₁₃t₂₃t₂₃/5760 − t₁₃t₁₃t₁₃t₂₃/1440,
0, 31 t₁₃t₂₃t₂₃t₂₃t₂₃t₂₃/967680 − 157 t₁₃t₁₃t₂₃t₂₃t₁₃t₂₃/1935360 − 31 t₁₃t₂₃t₁₃t₂₃t₂₃t₂₃/387072 − 31 t₁₃t₁₃t₂₃t₂₃t₂₃t₂₃/483840 + 11 t₁₃t₁₃t₁₃t₂₃t₁₃t₂₃/290304 + 31 t₁₃t₁₃t₂₃t₁₃t₂₃t₂₃/725760 + 83 t₁₃t₁₃t₁₃t₂₃t₂₃t₂₃/967680 − 13 t₁₃t₁₃t₁₃t₁₃t₂₃t₂₃/241920 + t₁₃t₁₃t₁₃t₁₃t₁₃t₂₃/60480, …]

The "buckle" Z_B, from Φ.



```
R = DKS[t[1, 2]/2];
Z_B = (-Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (-Φ)^σ[1,2,3] **
    Φ^σ[12,3,4];
Z_B@{4}
```

DKS[t₂₃/2, −1/12 t₁₃t₂₃ − 1/24 t₁₄t₂₄ + 1/24 t₁₄t₃₄ + 1/12 t₂₄t₃₄,
0, t₁₃t₂₃t₂₃t₂₃/5760 + 7 t₁₄t₂₄t₂₄t₂₄/5760 + t₁₄t₃₄t₂₄t₂₄/1920 − t₁₄t₃₄t₃₄t₂₄/1920 − 7 t₁₄t₃₄t₃₄t₃₄/5760 − t₂₄t₃₄t₃₄t₃₄/5760 + t₁₄t₂₄t₃₄t₂₄/1920 + t₁₄t₂₄t₁₄t₃₄/1920 − t₁₄t₃₄t₂₄t₃₄/1920 − 1/720 t₁₃t₁₃t₂₃t₂₃ + 1/720 t₁₃t₁₃t₁₃t₂₃ − 7 t₁₄t₁₄t₂₄t₂₄/5760 + 7 t₁₄t₁₄t₃₄t₃₄/5760 − t₁₄t₂₄t₃₄t₃₄/5760 + t₁₄t₁₄t₁₄t₂₄/1440 − t₁₄t₁₄t₁₄t₃₄/1440 − 1/960 t₁₄t₁₄t₂₄t₃₄ + t₁₄t₂₄t₂₄t₃₄/5760 − 1/960 t₂₄t₂₄t₃₄t₃₄ − t₂₄t₂₄t₂₄t₃₄/5760, …]

V from Z_B, following [AET, BND].

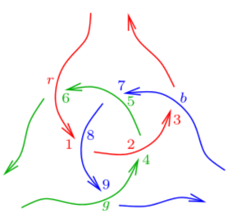
```
(El[Z_B // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη¹ // tη³ //
    hη² // hη⁴ // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])⟦
 1⟧
```

⟨1 → LS[0, −12/24, 0, 71112/5760 − 71122/5760 + 1222/1440, 0,
−3111 1112/967680 + 31111122/483840 − 83111222/967680 − 311112122/725760 − 311 11212/645120 + 131122 22/241920 + 101122 1222/1451520 + 52711 2212/5806080 − 122222/60480, …],
2 → LS[1/2, −12/12, 0, 1112/5760 − 1/720 1 122 + 1/720 122 2,
−11112/7680 + 11122/3840 − 11212/6912,
−111112/645120 + 23111122/483840 − 13111222/161280 − 112122/22680 − 4111212/580608 + 112222/15120 + 121222/12096 + 7111 2212/483840 − 122222/30240, …]⟩

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```

```
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ⟦1⟧_r@{5}, ξ⟦2⟧@{5}} // Print
```



;

{LS[0, bg, 1/2 bbg + bgr + 1/2 bgg,
1/6 b bbg + 1/2 b bgr + 1/2 b ggr + 1/4 b bgg + 1/2 b grr + 1/6 bggg,
1/24 bb bbg + 1/6 bb bgr + 1/4 bb ggr + 1/12 bb bgg +
1/4 bb grr + 1/6 bg ggr + 1/4 bg grr − b bgr g +
1/12 b bgg g − 2 b brg g + 1/6 b grr r + 1/2 bg bgr −
bg brg − 1/12 bbg bg − 1/2 bgr gr + 1/24 bgg gg, …],
CWS[0, 0, 2 bgr, bbgr − bgbr + bggr − bgrg + bgrr − brgr, bbbgr/3 −
bbgbr/2 + bbggr/2 + bbgrg/2 + bbgrr/2 + bbrbg/2 − 3 bbrgr/2 + bgbrr/2 − 3 bggbr/2 + bgggr/3 −
bggrg/2 + bggrr/2 + bgrgg/2 − 3 bgrrg/2 + bgrrr/3 + brggr/2 − brgrr/2 + brrgr/2, …]}

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yet×2.

Warning. Fidgety!

Braids and the Grothendieck−Teichmuller Group

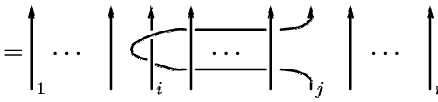
Dror Bar−Natan, the Newton Institute, January 2013, http://www.math.toronto.edu/~drorbn/Talks/Newton−1301/

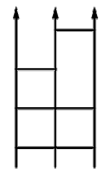
Abstract. The “Grothendieck-Teichmuller Group” (**GT**) appears as a “depth certificate” in many recent works — “we do A to B, apply the result to C, and get something related to **GT**, therefore it must be interesting”. Interesting or not, in my talk I will explain how **GT** arose first, in Drinfel’d’s work on associators, and how it can be used to show that “every bounded-degree associator extends”, that “rational associators exist”, and that “the pentagon implies the hexagon”⋆.

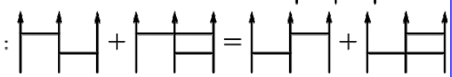
In a nutshell: the filtered tower of braid groups (with bells and whistles attached) is isomorphic to its associated graded, but the isomorphism is neither canonical nor unique — such an isomorphism is precisely the thing called “an associator”. But the set of isomorphisms between two isomorphic objects **always** has two groups acting simply transitively on it — the group of automorphisms of the first object acting on the right, and the group of automorphisms of the second object acting on the left. In the case of associators, that first group is what Drinfel’d calls the Grothendieck-Teichmuller group **GT**, and the second group, isomorphic but not canonically to the first and denoted **GRT**, is the one several recent works seem to refer to.

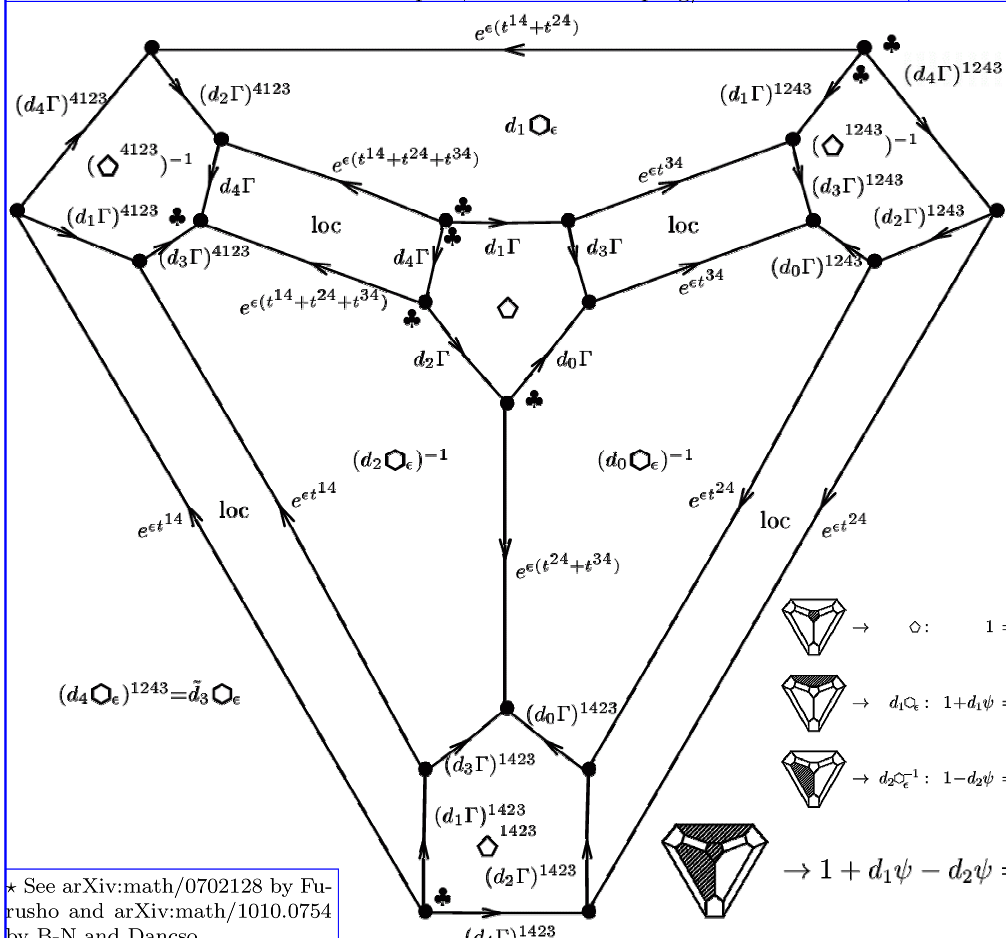
Almost everything I will talk about is in my old paper “On Associators and the Grothendieck-Teichmuller Group I”, also at arXiv:q-alg/9606021.

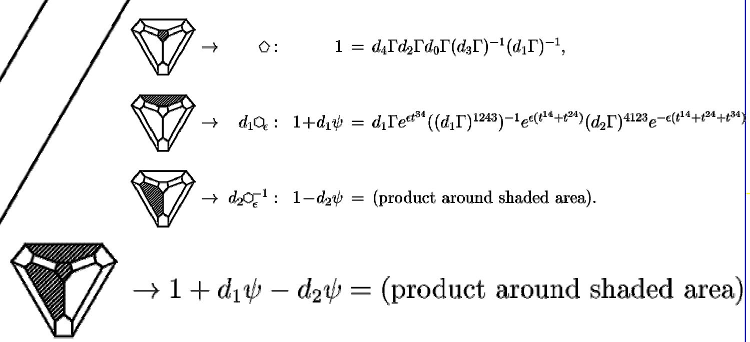
Baby(?) Example. PB_n: pure braids; $I \subset \mathbb{Q}PB_n$ the augmentation ideal; $B^{(m)} = \mathbb{Q}PB_n/I^{m+1}$ (filtered!); $\hat{B} = \varprojlim B^{(m)}$ (filtered!). Then $\operatorname{gr} B^{(m)} = C^{(m)}$ and then $\operatorname{gr}\hat{B} = \hat{C}$ where $C = \langle t^{ij} = t^{ji} \colon [t^{ij}, t^{kl}] = [t^{ij}, t^{ik} + t^{jk}] = 0\rangle$, so $B^{(m)}$ and $\hat{B}$ are isomorphic to $C^{(m)}$ and $\hat{C}$, but not canonically. Me not know that the groups **GT** and **GRT** here have been analyzed.

$\sigma^{ij} =$ 

$t^{13}t^{13}t^{12}t^{23} \longleftrightarrow$ 

$4T:$ 





⋆ See arXiv:math/0702128 by Furusho and arXiv:math/1010.0754 by B-N and Dancso.

Main Theorem. The projection $\mathbf{ASSO}^{(m)} \to \mathbf{ASSO}^{(m-1)}$ is surjective. (yey!)

Sketch. Given $\mathbf{ASSO}^{(m)} \neq \emptyset$ (hard, analytic), sufficient is surjectivity of $\mathbf{GRT}^{(m)} \to \mathbf{GRT}^{(m-1)}$, enough is surjectivity of $\mathfrak{grt}^{(m)} \to \mathfrak{grt}^{(m-1)}$, polyhedron on left use, little homological algebra too.

by successive approximations presents no problems. For this we introduce the following modification $\mathrm{GRT}(k)$ of the group $\mathrm{GT}(k)$. We denote by $\mathrm{GRT}_1(k)$ the set of all $g \in \mathrm{Fr}_k(A, B)$ such that

$$g(B, A) = g(A, B)^{-1}, \tag{5.12}$$

$$g(C, A)g(B, C)g(A, B) = 1 \quad \text{for } A + B + C = 0, \tag{5.13}$$

$$A + g(A, B)^{-1}Bg(A, B) + g(A, C)^{-1}Cg(A, C) = 0 \quad \text{for } A + B + C = 0, \tag{5.14}$$

$$g(X^{12}, X^{23} + X^{24})g(X^{13} + X^{23}, X^{34}) = g(X^{23}, X^{34})g(X^{12} + X^{13}, X^{24} + X^{34})g(X^{12}, X^{23}), \tag{5.15}$$

where the X^{ij} satisfy (5.1). $\mathrm{GRT}_1(k)$ is a group with the operation

$$(g_1 \circ g_2)(A, B) = g_1(g_2(A, B)Ag_2(A, B)^{-1}, B) \cdot g_2(A, B). \tag{5.16}$$

On $\mathrm{GRT}_1(k)$ there is an action of k^*, given by $\tilde{g}(A, B) = g(c^{-1}A, c^{-1}B)$, $c \in k^*$. The semidirect product of k^* and $\mathrm{GRT}_1(k)$ we denote by $\mathrm{GRT}(k)$. The Lie algebra $\mathfrak{grt}_1(k)$ of the group $\mathrm{GRT}_1(k)$ consists of the series $\psi \in \mathfrak{fr}_k(A, B)$ such that

$$\psi(B, A) = -\psi(A, B), \tag{5.17}$$

$$\psi(C, A) + \psi(B, C) + \psi(A, B) = 0 \quad \text{for } A + B + C = 0, \tag{5.18}$$

$$[B, \psi(A, B)] + [C, \psi(A, C)] = 0 \quad \text{for } A + B + C = 0, \tag{5.19}$$

$$\psi(X^{12}, X^{23} + X^{24}) + \psi(X^{13} + X^{23}, X^{34}) = \psi(X^{23}, X^{34}) + \psi(X^{12} + X^{13}, X^{24} + X^{34}) + \psi(X^{12}, X^{23}), \tag{5.20}$$

where the X^{ij} satisfy (5.1). A commutator $\langle\ ,\ \rangle$ in $\mathfrak{grt}_1(k)$ is of the form

$$\langle\psi_1, \psi_2\rangle = [\psi_1, \psi_2] + D_{\psi_2}(\psi_1) - D_{\psi_1}(\psi_2), \tag{5.21}$$

where $[\psi_1, \psi_2]$ is the commutator in $\mathfrak{fr}_k(A, B)$ and D_ψ is the derivation of $\mathfrak{fr}_k(A, B)$ given by $D_\psi(A) = [\psi, A]$, $D_\psi(B) = 0$. The algebra $\mathfrak{grt}_1(k)$ is

Proposition 5.1. *The action of* $\mathrm{GT}(k)$ *on* $M(k)$ *is free and transitive.*

Proof. If $(\mu, \varphi) \in M(k)$ and $(\overline{\mu}, \overline{\varphi}) \in M(k)$, then there is exactly one f such that $\overline{\varphi}(A, B) = f(\varphi(A, B)e^A\varphi(A, B)^{-1}, e^B) \cdot \varphi(A, B)$. We need to show that $(\lambda, f) \in \mathrm{GT}(k)$, where $\lambda = \overline{\mu}/\mu$. We prove (4.10). Let G_n be the semidirect product of S_n and $\exp\mathfrak{a}_n^k$. Consider the homomorphism $B_n \to G_n$ that takes σ_i into

$$\varphi(X^{1i} + \cdots + X^{i-1,i}, X^{i,i+1})^{-1}\sigma^{i,i+1}e^{\mu X^{i,i+1}/2}\varphi(X^{1i} + \cdots + X^{i-1,i}, X^{i,i+1}),$$

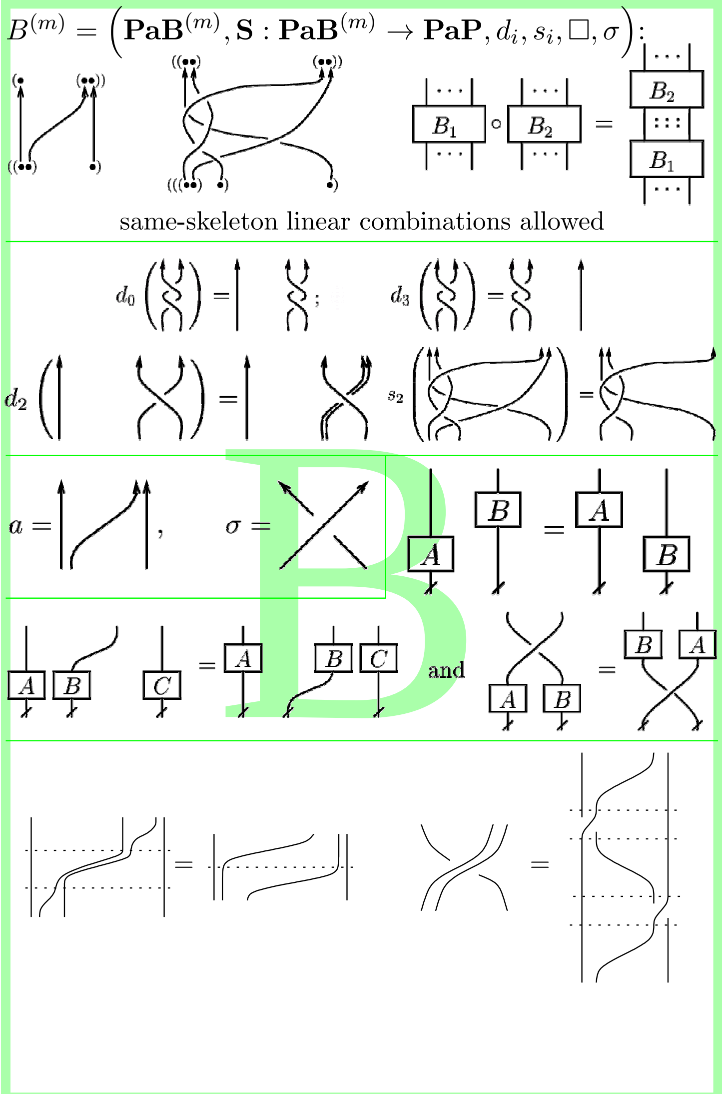
where $\sigma^{ij} \in S_n$ transposes i and j. It induces a homomorphism $K_n \to \exp\mathfrak{a}_n^k$, and therefore a homomorphism $\alpha_n\colon K_n(k) \to \exp\mathfrak{a}_n^k$, where $K_n(k)$ is the k-pro-unipotent completion of K_n. It is easily shown that the left- and right-hand sides of (4.10) have the same images in $\exp\mathfrak{a}_4^k$. It remains to prove that α_n is an isomorphism. The algebra Lie $K_n(k)$ is topologically generated by the elements ξ_{ij}, $1 \le i < j \le n$, with defining relations obtained from (4.7)–(4.9) by substituting $x_{ij} = \exp\xi_{ij}$. The principal parts of these relations are the same as in (5.1), while $(\alpha_n)_*(\xi_{ij}) = \mu X^{ij} +$ {lower terms}, where $(\alpha_n)_*$: Lie $K_n(k) \to \mathfrak{a}_n^k$ is induced by the homomorphism α_n. Therefore α_n is an isomorphism, i.e., (4.10) is proved. (4.3) is obvious. To prove (4.4), we can interpret it in terms of K_3 and argue as in the proof of (4.10), or, what is equivalent, make the substitution

$$X_1 = e^A, \quad X_2 = e^{-A/2}\varphi(B, A)e^B\varphi(B, A)^{-1}e^{A/2}, \quad X_3 = \varphi(C, A)e^C\varphi(C, A)^{-1}, \tag{5.4}$$

where $A + B + C = 0$. ●

From Drinfel’d’s *On quasitriangular Quasi-Hopf algebras and a group closely connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

The Main Course



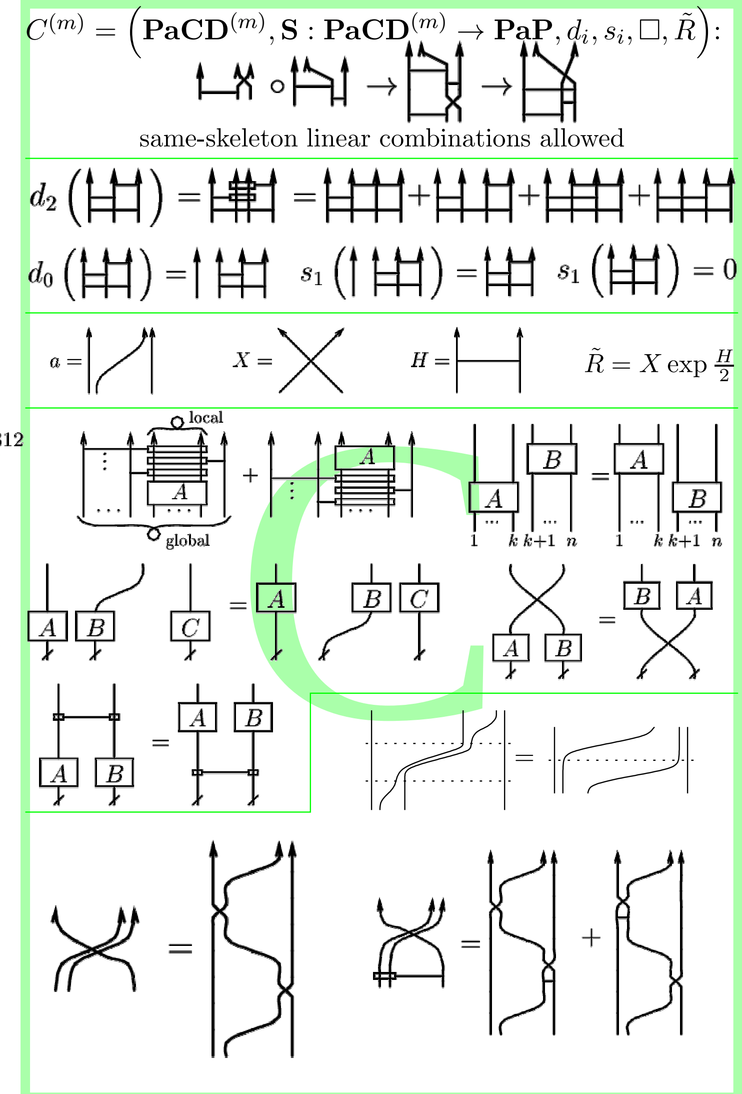
$$d_4\Phi \cdot d_2\Phi \cdot d_0\Phi = d_1\Phi \cdot d_3\Phi$$

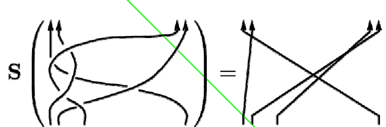
$$d_1 \exp\left(\pm\frac{1}{2}t^{12}\right) =$$

$$\Phi \cdot \exp\left(\pm\frac{1}{2}t^{23}\right) \cdot (\Phi^{-1})^{132} \cdot \exp\left(\pm\frac{1}{2}t^{13}\right) \cdot \Phi^{312}$$

$$s_1\Phi = s_2\Phi = s_3\Phi = 1$$

$$\square\Phi = \Phi \otimes \Phi$$





$$d_4\Gamma \cdot d_2\Gamma \cdot d_0\Gamma = d_1\Gamma \cdot d_3\Gamma$$

$$1 = \Gamma \cdot (\Gamma^{-1})^{132} \cdot \Gamma^{312}$$

$$d_1 t^{12} = \Gamma \cdot \left(t^{23} \cdot (\Gamma^{-1})^{132} + (\Gamma^{-1})^{132} \cdot t^{13}\right) \cdot \Gamma^{312}$$

$$e^{\epsilon(t^{13}+t^{23})} = \Gamma \cdot e^{\epsilon t^{23}} \cdot (\Gamma^{-1})^{132} \cdot e^{\epsilon t^{13}} \cdot \Gamma^{312}$$

I have a nifty
Free Lie calculator.
Wanna play?

A Partial Reduction of BF Theory to Combinatorics, 1

Abstract. I will describe a semi-rigorous reduction of perturbative BF theory (Cattaneo-Rossi [CR]) to computable combinatorics, in the case of ribbon 2-links. Also, I will explain how and why my approach may or may not work in the non-ribbon case. Weak this result is, and at least partially already known (Watanabe [Wa]). Yet in the ribbon case, the resulting invariant is a universal finite type invariant, a gadget that significantly generalizes and clarifies the Alexander polynomial and that is closely related to the Kashiwara-Vergne problem. I cannot rule out the possibility that the corresponding gadget in the non-ribbon case will be as interesting. (good news in highlight)

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then for a 2-link $(f_t)_{t\in T}$,


Cattaneo

Rossi

$$\zeta = \log \sum_{\substack{\text{diagrams}\\ D}} \frac{D}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

is an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

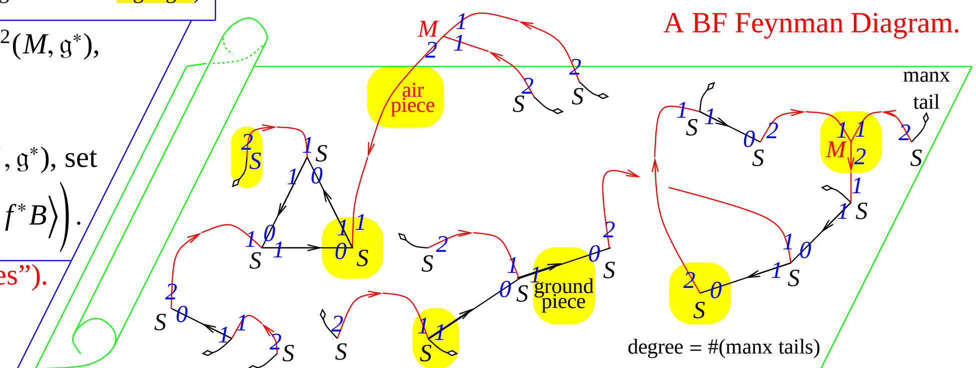
BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A,B) := \int_M \langle B, F_A \rangle.$$

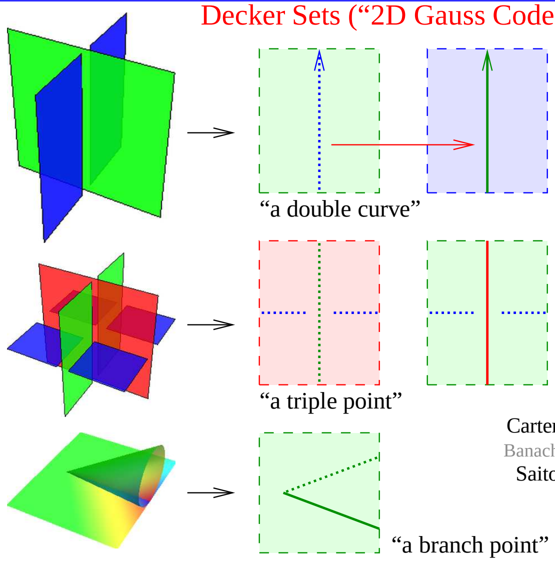
With $f\colon (S = \mathbb{R}^2) \to M$, $\xi \in \Omega^0(S, \mathfrak{g})$, $\beta \in \Omega^1(S, \mathfrak{g}^*)$, set

$$\mathcal{O}(A,B,f) := \int \mathcal{D}\xi\mathcal{D}\beta \exp\left(\frac{i}{\hbar}\int_S \left\langle \xi, d_{f^*A}\beta + f^*B \right\rangle\right).$$

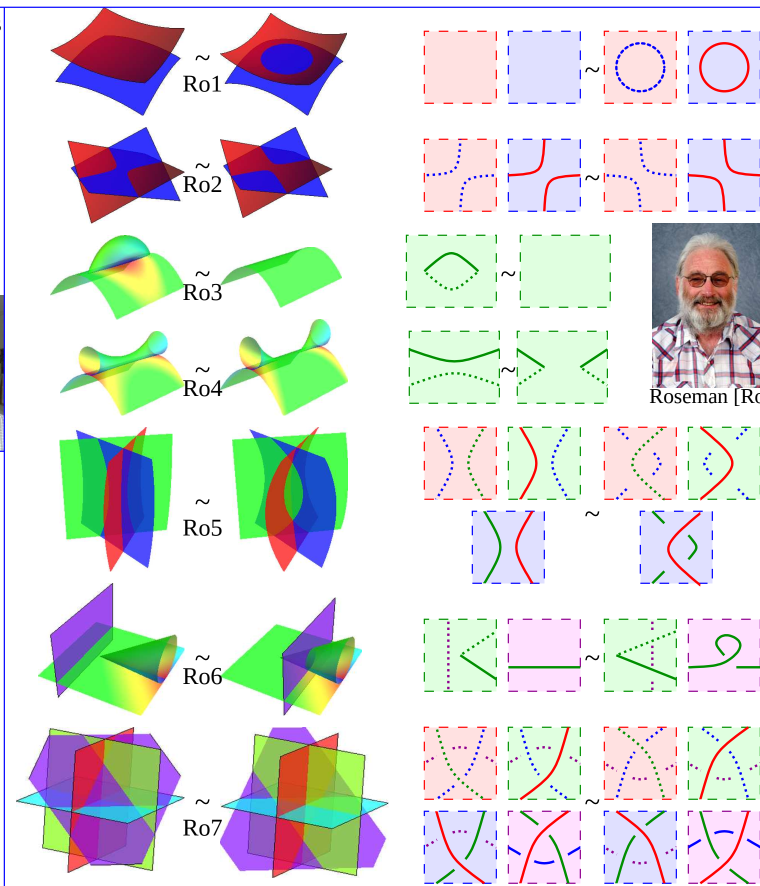
A BF Feynman Diagram.



Decker Sets ("2D Gauss Codes").

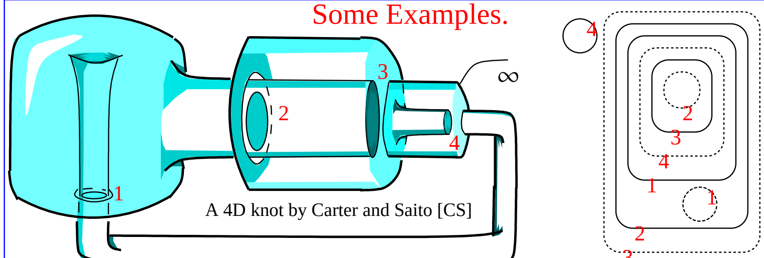



Carter
Banach
Saito

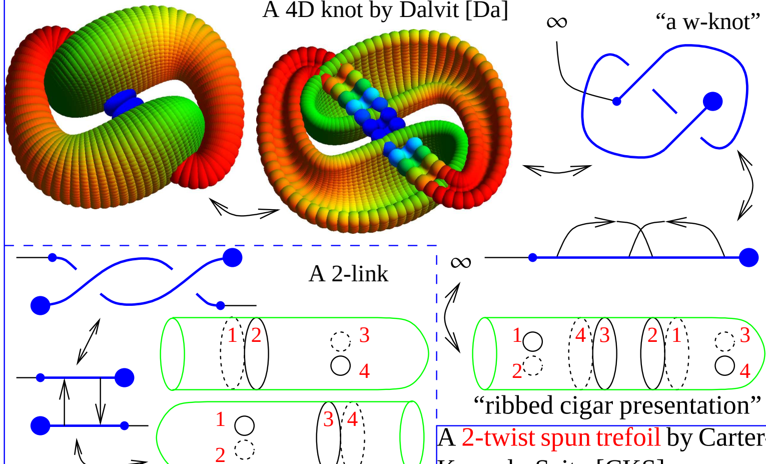


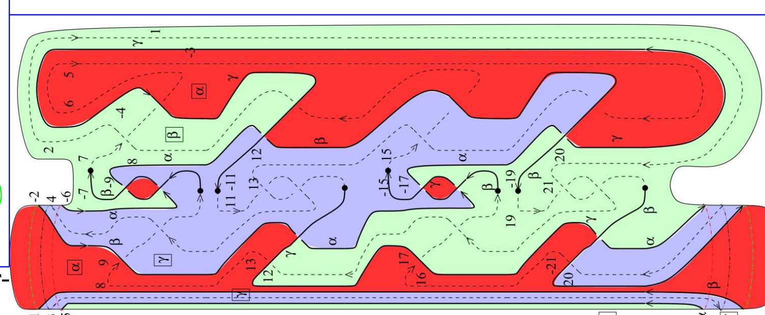
Roseman [Ro]

Some Examples.


A 4D knot by Carter and Saito [CS]

A 4D knot by Dalvit [Da]

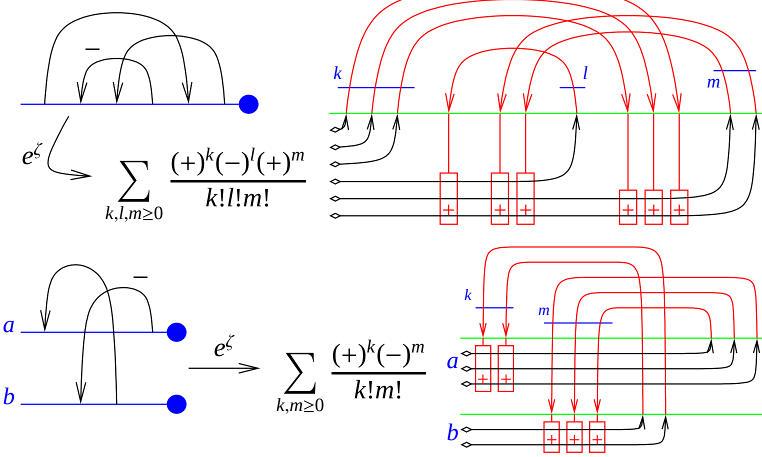


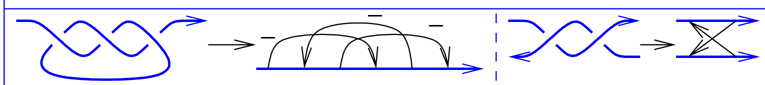


A 2-twist spun trefoil by Carter-Kamada-Saito [CKS].

A Partial Reduction of BF Theory to Combinatorics, 2

Theorem 1 (with Cattaneo (credit, no blame)). In the ribbon case, e^ζ can be computed as follows:





Theorem 2. Using Gauss diagrams to represent knots and T-component pure tangles, the above formulas define an invariant in $CW(FL(T)) \to CW(T)$, "cyclic words in T".

• Agrees with BN-Dancso [BND] and with [BN2]. • In-practice computable! • Vanishes on braids. • Extends to w. • Contains Alexander. • The "missing factor" in Levine's factorization [Le] (the rest of [Le] also fits, hence contains the MVA). • Related to / extends Farber's [Fa]? • Should be summed and categorified.

References.

[Ar] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series **5**, American Mathematical Society 1994.

[BN1] D. Bar-Natan, *Bracelets and the Goussarov filtration of the space of knots, Invariants of knots and 3-manifolds (Kyoto 2001)*, Geometry and Topology Monographs **4** 1–12, arXiv:math.GT/0111267.

[BN2] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, http://www.math.toronto.edu/~drorbn/papers/KBH/, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne*, http://www.math.toronto.edu/~drorbn/papers/WKO/.

[CKS] J. S. Carter, S. Kamada, and M. Saito, *Diagrammatic Computations for Quandles and Cocycle Knot Invariants*, Contemp. Math. **318** (2003) 51–74.

[CS] J. S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs **55**, American Mathematical Society, Providence 1998.

[Da] E. Dalvit, http://science.unitn.it/~dalvit/.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[Fa] M. Farber, *Noncommutative Rational Functions and Boundary Links*, Math. Ann. **293** (1992) 543–568.

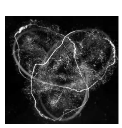
[Le] J. Levine, *A Factorization of the Conway Polynomial*, Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.

[Ro] D. Roseman, *Reidemeister-Type Moves for Surfaces in Four-Dimensional Space*, Knot Theory, Banach Center Publications **42** (1998) 347–380.

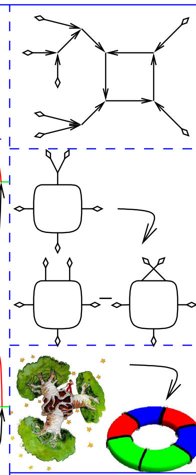
[Wa] T. Watanabe, *Configuration Space Integrals for Long n-Knots, the Alexander Polynomial and Knot Space Cohomology*, Alg. and Geom. Top. **7** (2007) 47-92, arXiv:math/0609742.

Continuing Joost Slingerland…

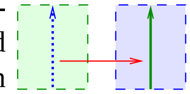
http://youtu.be/YCA0VIExVhge

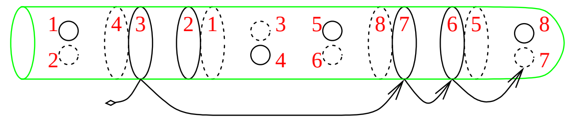


http://youtu.be/mHyTOcfF99o



Sketch of Proof. In 4D axial gauge, only "drop down" red propagators, hence in the ribbon case, no M-trivalent vertices. S integrals are ± 1 iff "ground pieces" run on nested curves as below, and exponentials arise when several propagators compete for the same double curve. And then the combinatorics is obvious…

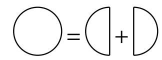


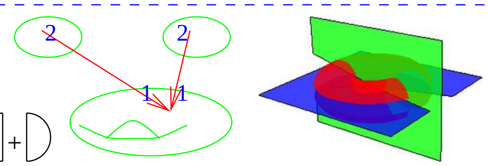


Musings

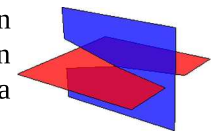
Chern-Simons. When the domain of BF is restricted to ribbon knots, and the target of CS is restricted to trees and wheels, they agree. Why?

Is this all? What about the ∨-invariant? (the "true" triple linking number)

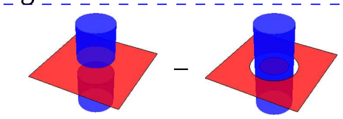




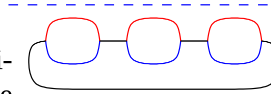
Gnots. In 3D, a generic immersion of S^1 is an embedding, a knot. In 4D, a generic immersion of a surface has finitely-many double points (a gnot?). Perhaps we should be studying these?



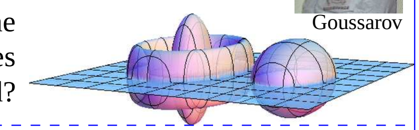
Finite type. What are finite-type invariants for 2-knots? What would be "chord diagrams"?



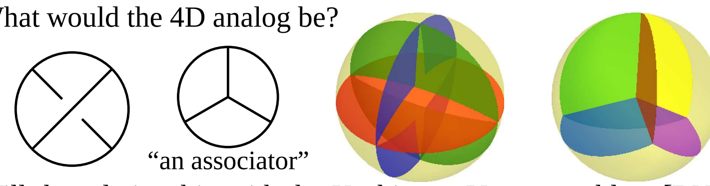
Bubble-wrap-finite-type. There's an alternative definition of finite type in 3D, due to Goussarov (see [BN1]). The obvious parallel in 4D involves "bubble wraps". Is it any good?



Goussarov



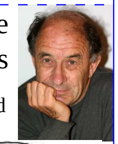
Shielded tangles. In 3D, one can't zoom in and compute "the Chern-Simons invariant of a tangle". Yet there are well-defined invariants of "shielded tangles", and rules for their compositions. What would the 4D analog be?



"an associator"

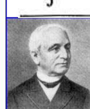
Will the relationship with the Kashiwara-Vergne problem [BND] necessarily arise here?

Plane curves. Shouldn't we understand integral / finite type invariants of plane curves, in the style of Arnold's J^+, J^-, and St [Ar], a bit better?



Arnold

	a(·)	a(·)	a(·)	∞	○	◎	◎◎	◎◎◎	···
St	1	0	0	0	0	1	2	3	···
J^+	0	2	0	0	0	−2	−4	−6	···
J^-	0	0	−2	−1	0	−3	−6	−9	···



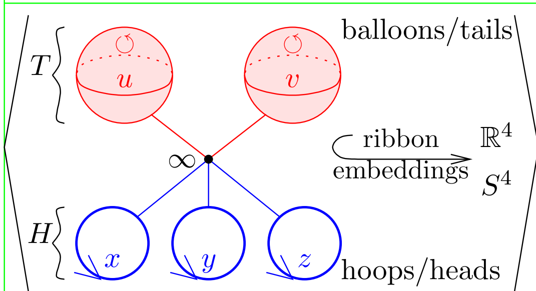
"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Finite Type Invariants of Ribbon Knotted Balloons and Hoops

Abstract. On my September 17 Geneva talk (ω/sep) I described a certain trees-and-wheels-valued invariant ζ of ribbon knotted loops and 2-spheres in 4-space, and my October 8 Geneva talk (ω/oct) describes its reduction to the Alexander polynomial. Today I will explain how that same invariant arises completely naturally within the theory of finite type invariants of ribbon knotted loops and 2-spheres in 4-space.

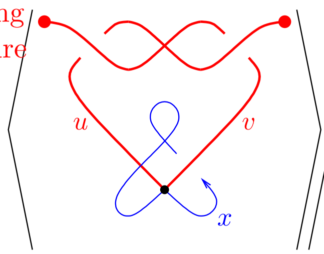


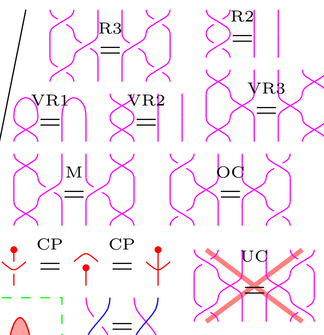
My goal is to tell you why such an invariant is expected, yet not to derive the computable formulas.

Disturbing Conjecture

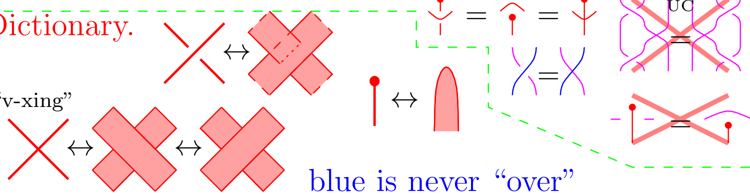
$\mathcal{K}^{bh} = \mathbb{Q}$

Satoh





Dictionary.



Expansions

the semi-virtual 

Let $\mathcal{I}^n := \langle$pictures with $\geq n$ semi-virts$\rangle \subset \mathcal{K}^{bh}$.
We seek an "expansion"

$$Z\colon \mathcal{K}^{bh} \to \operatorname{gr}\mathcal{K}^{bh} = \widehat{\bigoplus} \mathcal{I}^n/\mathcal{I}^{n+1} =: \mathcal{A}^{bh}$$

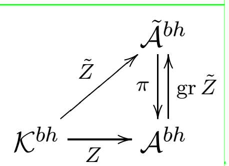
satisfying "property U": if $\gamma \in \mathcal{I}^n$, then

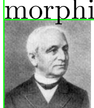
$$Z(\gamma) = (0,\dots,0,\gamma/\mathcal{I}^{n+1},*,*,\dots).$$

X.-S. Lin

Why? • Just because, and this is vastly more general.
• $\left(\mathcal{K}^{bh}/\mathcal{I}^{n+1}\right)^\star$ is "finite-type/polynomial invariants".
• The Taylor example: Take $\mathcal{K} = C^\infty(\mathbb{R}^n)$, $\mathcal{I} = \{f \in \mathcal{K}\colon f(0)=0\}$. Then $\mathcal{I}^n = \{f\colon f$ vanishes like $|x|^n\}$ so $\mathcal{I}^n/\mathcal{I}^{n+1}$ is homogeneous polynomials of degree n and Z is a "Taylor expansion"! (So Taylor expansions are vastly more general than you'd think).

Plan. We'll construct a graded $\tilde{\mathcal{A}}^{bh}$, a surjective graded $\pi\colon \tilde{\mathcal{A}}^{bh} \to \mathcal{A}^{bh}$, and a filtered $\tilde{Z}\colon \mathcal{K}^{bh} \to \mathcal{A}^{bh}$ so that $\pi \,/\!\!/\, \operatorname{gr}\tilde{Z} = Id$ (property U: if $\deg D = n$, $\tilde{Z}(\pi(D)) = \pi(D) + (\deg \geq n)$). Hence • π is an isomorphism. • $Z := \tilde{Z} \,/\!\!/\, \pi$ is an expansion.

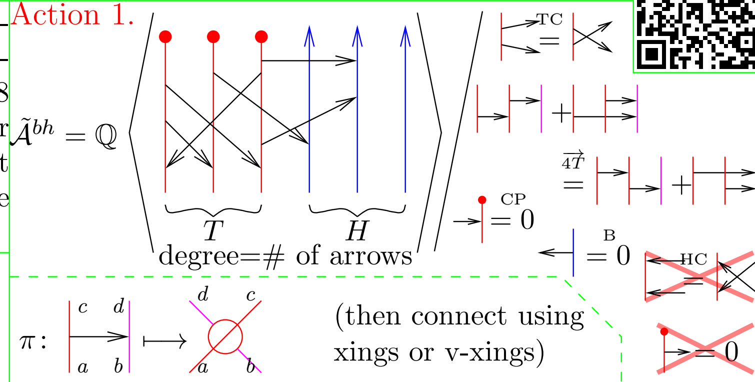




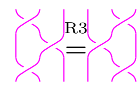
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

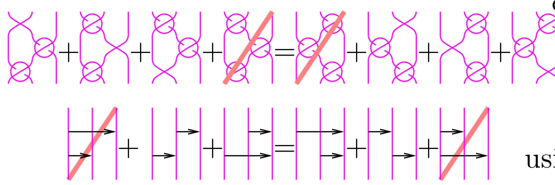
Action 1.



(then connect using xings or v-xings)

Deriving $\overrightarrow{4T}$. Start from  key: use

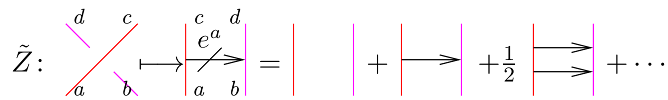
Goussarov-Polyak-Viro



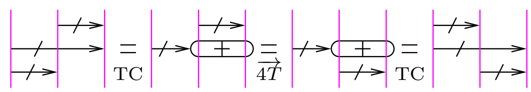
in $\mathcal{I}^n/\mathcal{I}^{n+1}$

using TC □

Action 2.

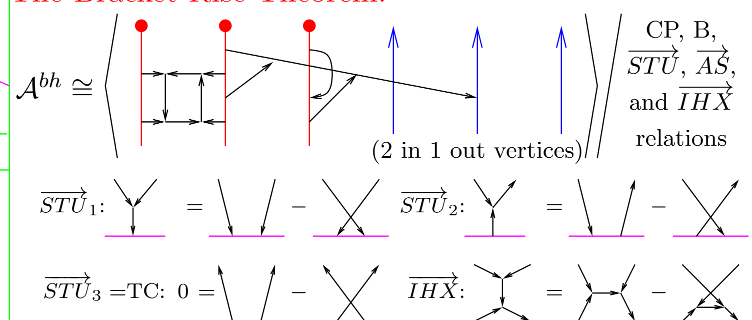


R3.

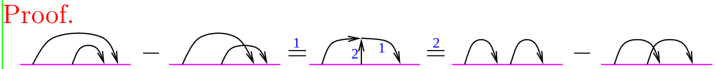


Exercise. Prove property U.

The Bracket-Rise Theorem.



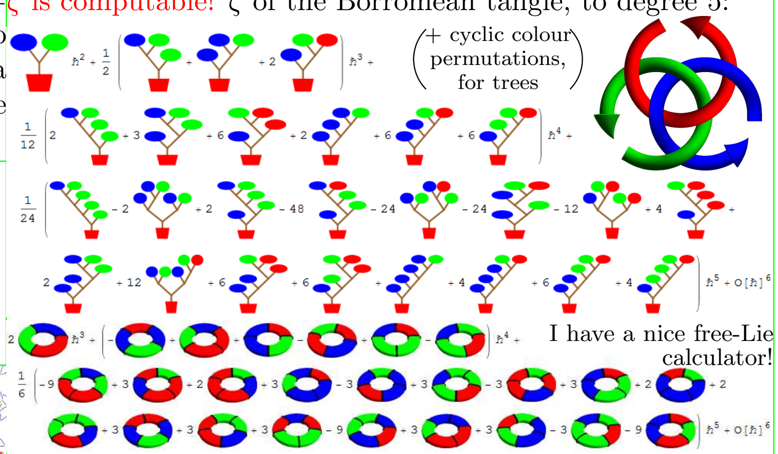
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only trees and wheels persist.

Theorem. $\mathcal{A}^{bh}$ is a bi-algebra. The space of its primitives is $FL(T)^H \times CW(T)$, and $\zeta = \log Z$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



I have a nice free-Lie calculator!

The Kashiwara-Vergne Problem and Topology

ω :=http://www.math.toronto.edu/~drorbn/Talks/Bern-131104
Video, handout, links at ω/

Abstract. I will describe a general machine, a close cousin of Taylor's theorem, whose inputs are topics in topology and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that

$$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F+(e^{\operatorname{ad} y}-1)G \text{ in } FL$$

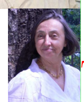
and so that with $z=\log e^x e^y$,

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F+\operatorname{tr}(\operatorname{ad} y)\partial_y G \quad \text{in cyclic words}$$

$$=\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1}+\frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1}-\frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1}-1\right)$$

Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.


Kashiwara


Vergne


Alekseev


Meinrenken


Torossian

The Machine. Let G be a group, $\mathcal{K}=\mathbb{Q}G=\{\sum a_i g_i\colon a_i\in\mathbb{Q},\, g_i\in G\}$ its group-ring, $\mathcal{I}=\{\sum a_i g_i\colon \sum a_i=0\}\subset\mathcal{K}$ its augmentation ideal. Let

$$\mathcal{A}=\widehat{\operatorname{gr}\mathcal{K}}:=\widehat{\bigoplus}_{m\geq 0}\mathcal{I}^m/\mathcal{I}^{m+1}.$$

P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon\mathcal{K}\to\mathcal{A}$ is an "expansion" if for any $\gamma\in\mathcal{I}^m$, $Z(\gamma)=(0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, and a "homomorphic expansion" if in addition it preserves the product.

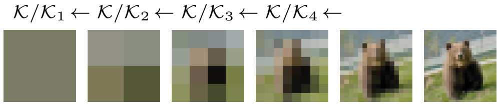
Example. Let $\mathcal{K}=C^\infty(\mathbb{R}^n)$ and $\mathcal{I}=\{f\colon f(0)=0\}$. Then $\mathcal{I}^m=\{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A}=\{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun. $\mathcal{K}=\left\{\right.$  $\left.\right\}=$ (The set of all 2D projections of reality)

ω/Bear

$\mathcal{K}/\mathcal{K}_1\leftarrow\mathcal{K}/\mathcal{K}_2\leftarrow\mathcal{K}/\mathcal{K}_3\leftarrow\mathcal{K}/\mathcal{K}_4\leftarrow$ ...

Crop Rotate Adjoin



An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1\oplus\mathcal{K}_1/\mathcal{K}_2\oplus\mathcal{K}_2/\mathcal{K}_3\oplus\mathcal{K}_3/\mathcal{K}_4\oplus\mathcal{K}_4/\mathcal{K}_5\oplus\mathcal{K}_5/\mathcal{K}_6\oplus\ \cdots$

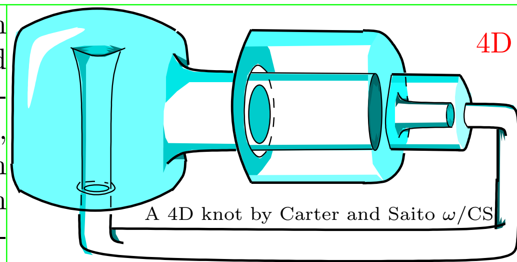
$\mathcal{K}/\mathcal{K}_1=\mathbb{R}^3$, $\mathcal{K}_3/\mathcal{K}_4=\ker(\mathcal{K}/\mathcal{K}_4\to\mathcal{K}/\mathcal{K}_3)$

In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

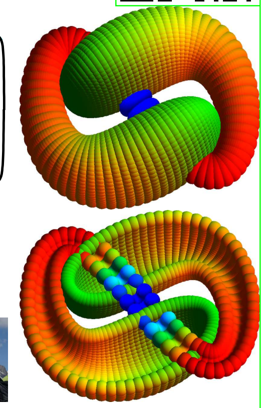
Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg.


Dancso, ω/ZD

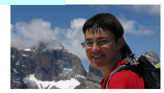
4D Knots.


A 4D knot by Carter and Saito ω/CS

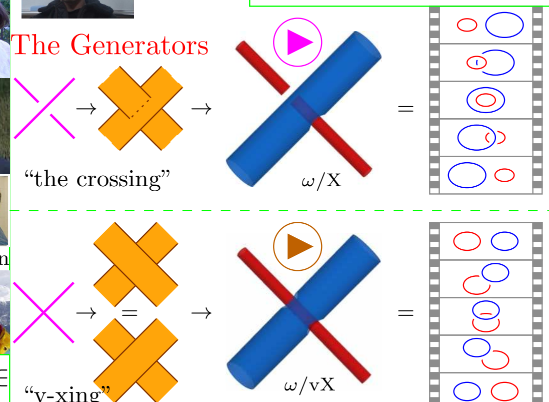

ω/F

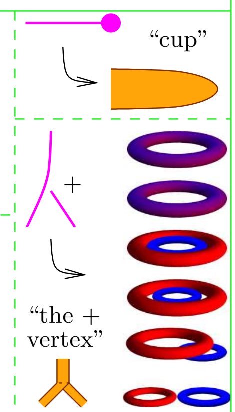



Satoh

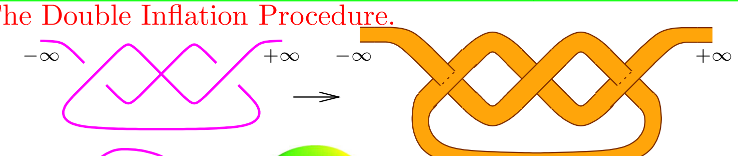

Dalvit ω/Dal

The Generators

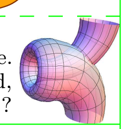




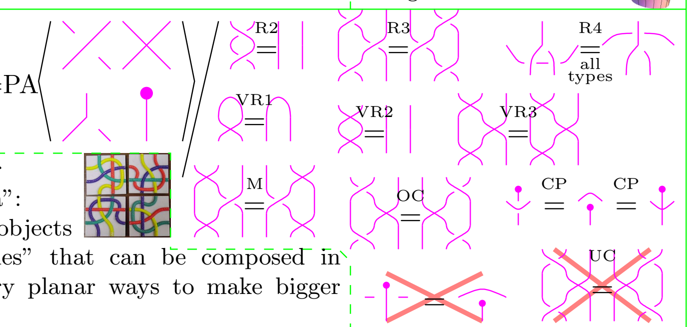
The Double Inflation Procedure.



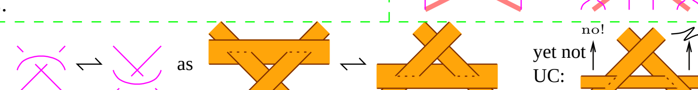
Riddle. What band, inflated, gives the "Wen"?



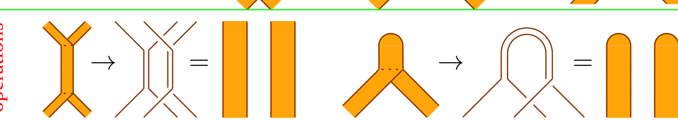
wKO. $w\mathcal{K}:=\mathrm{PA}\langle\ \ \rangle$



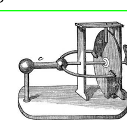
"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles.



The unary w-operations



The Machine generalizes to arbitrary algebraic structures!


ω/mac

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

1. $\mathrm{proj}\,\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$

— All Signs Are Wrong! —

I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

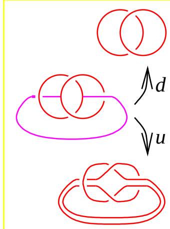
Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

Cans and Can't Yets.

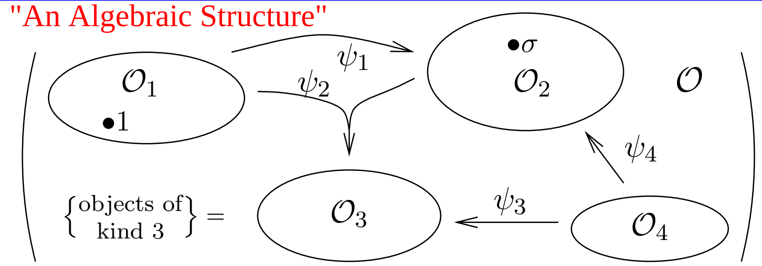
(arbitrary algebraic structure) $\xrightarrow[\text{machine}]{\text{projectivization}}$ (a problem in graded algebra)

www.katlas.org

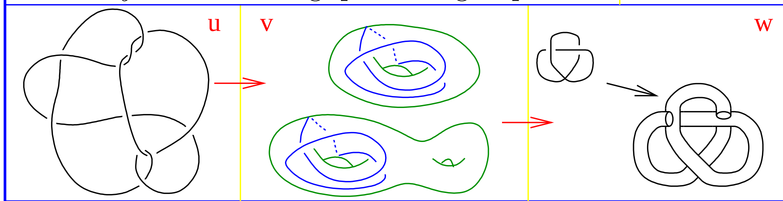
- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel'd associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.



"An Algebraic Structure"



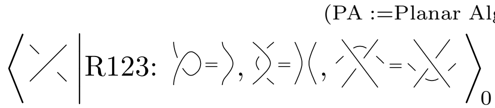
- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.



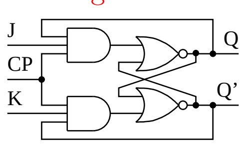
u-Knots

(PA :=Planar Algebra)

{knots &links} = PA ⟨ crossings | R123: … ⟩ (0 legs)



Circuit Algebras



A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

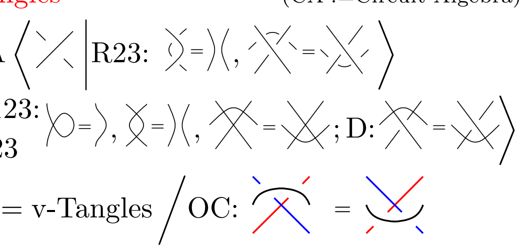
v-Tangles and w-Tangles

(CA :=Circuit Algebra)

{v-knots & links} = CA ⟨ crossing | R23: … ⟩

= PA ⟨ crossings, virtual crossing | VR123: …, R23: …; D: … ⟩

{w-Tangles} = v-Tangles / OC: …



Homomorphic expansions

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\mathrm{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

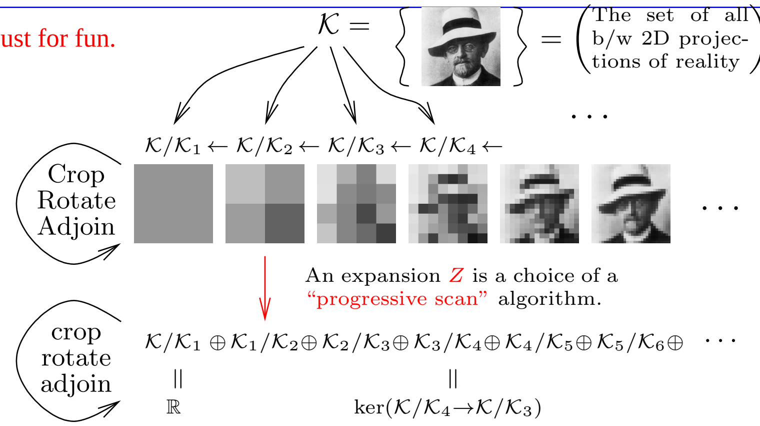
$$\Downarrow \qquad \downarrow Z$$

$$\mathrm{ops} \circlearrowright \mathrm{gr}\,\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

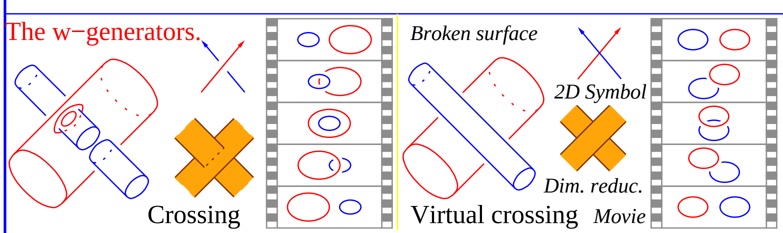
An expansion is a filtered $Z : \mathcal{K} \to \mathrm{gr}\,\mathcal{K}$ that "covers" the identity on $\mathrm{gr}\,\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Reality. $\mathrm{gr}\,\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded "guess" $\mathcal{A}$ with a surjection $\tau : \mathcal{A} \to \mathrm{gr}\,\mathcal{K}$ and a filtered $Z : \mathcal{K} \to \mathcal{A}$ for which $(\mathrm{gr}\,Z) \circ \tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for "homomorphic".

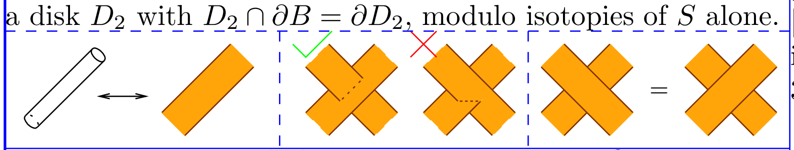
Just for fun.



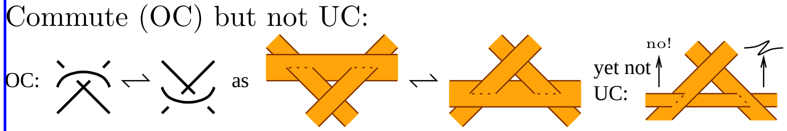
The w−generators.



A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



Filtered algebraic structures are cheap and plenty.

In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products"). In this case, set $\mathrm{proj}\,\mathcal{K} := \mathrm{gr}\,\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

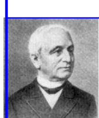
2. Pure braids — PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \mathrm{gr}\,PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

3. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathrm{proj}\,Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$



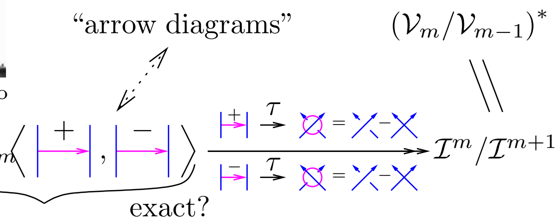
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

1. $\mathrm{proj}\,\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$, continued.

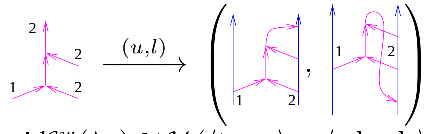

Goussarov-Polyak-Viro



Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \xrightarrow{\iota} \mathcal{PA}^w(\uparrow_n) \underset{l}{\overset{u}{\underset{\pi}{\rightleftarrows}}} \langle\text{trees}\rangle \longrightarrow 0,$$

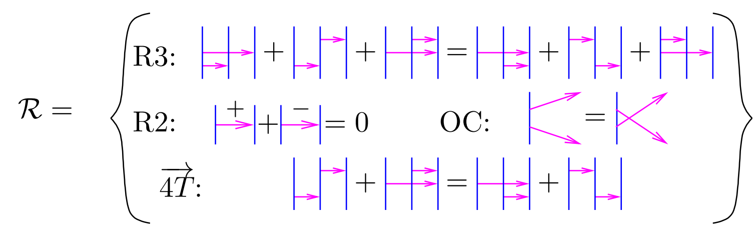
with



So $\mathrm{proj}\,\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}\left(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle\right)$.


trees atop a wheel, and a little prince.

Imperfect Thumb-Rule. Take R3 (say), substitute ⤧ → ⤧ + ⤧, keep the lowest degree terms that don't immediately die:

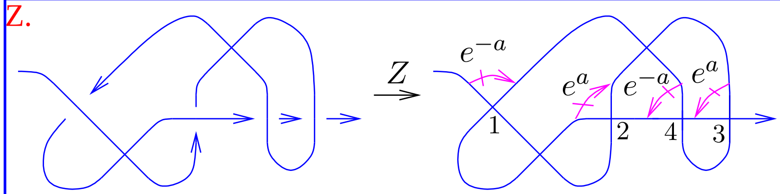


Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1,\dots,x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr}: \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

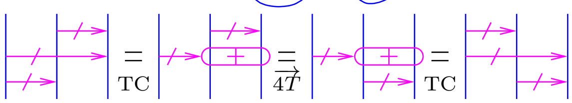
are "tangential derivations", so $D \leftrightarrow (a_1,\dots,a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\mathrm{div}: \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1,\dots,a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j: \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \mathrm{div}\,D$.

Z.



Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\mathrm{div}\,D = \iota^{-1}(u-l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

R3.



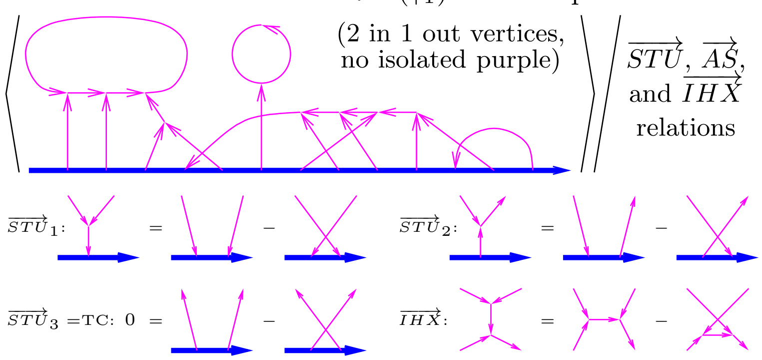
Thm. $Z \succ A$, maybe $\succ\!\succ$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\mathrm{div}\,D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD}\,\mathrm{div}\,D$.

The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to

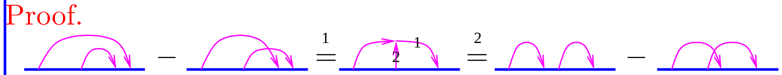


Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.

Theorem. $\mathfrak{sder}_n = \pi\alpha(\mathrm{proj}\,\text{u-tangles})$, where α is the obvious map $\mathrm{proj}\,\text{u-tangles} \to \mathrm{proj}\,\text{w-tangles}$.

Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

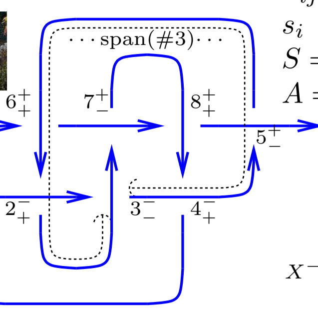
Proof.



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

The Alexander Theorem.


Sanderson

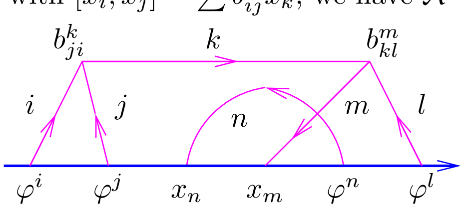


$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0\\ 0&0&1&0&1&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&1&0&0&1&0&1&0\\ 0&1&0&1&0&1&1&1\\ 0&1&0&1&0&0&1&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$$X^{-S} = \mathrm{diag}\left(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}\right).$$

To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij} x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via




Penrose Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij} b^m_{kl} \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

Conjecture. For u-knots, A is the Alexander polynomial.

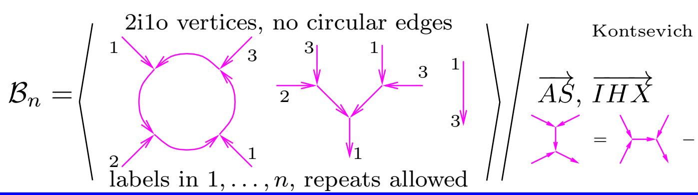
Theorem. With $w: x^k \mapsto w_k = $ (the k-wheel),

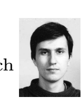
$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$,
$Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where




Kontsevich

2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan

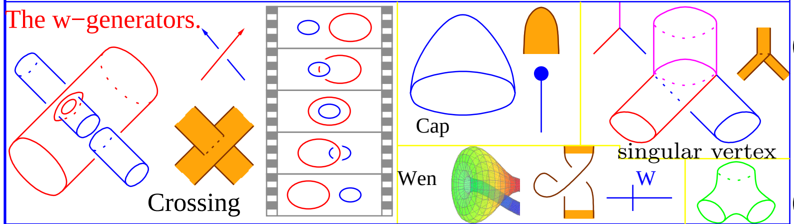
I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

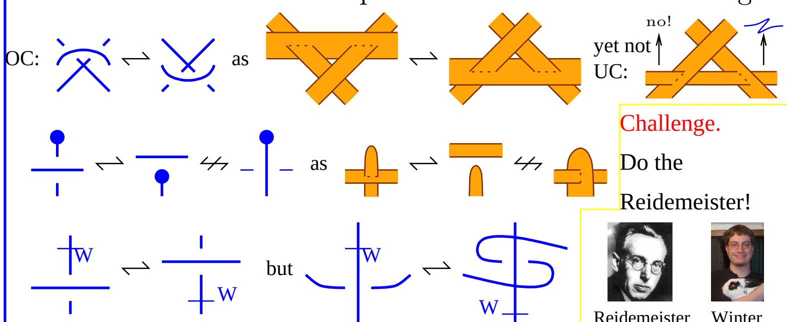
Trivalent w-Tangles.

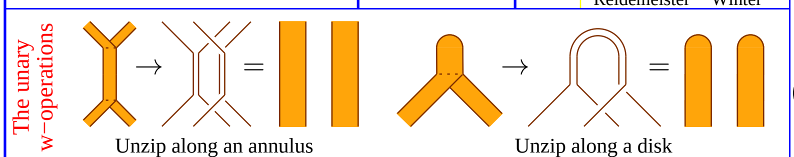
$$\text{wTT} = \text{CA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array} \middle| \begin{array}{c}\text{w-}\\ \text{relations}\end{array} \middle| \begin{array}{c}\text{unary w-}\\ \text{operations}\end{array} \right\rangle$$

The w−generators.



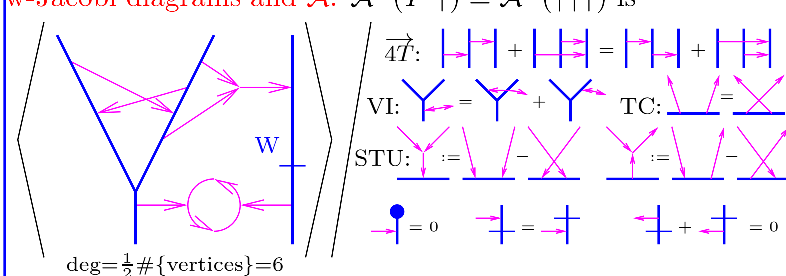
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:





w-Jacobi diagrams and $\mathcal{A}$.

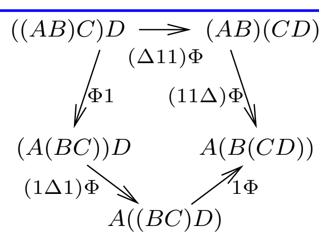
$\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



An Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

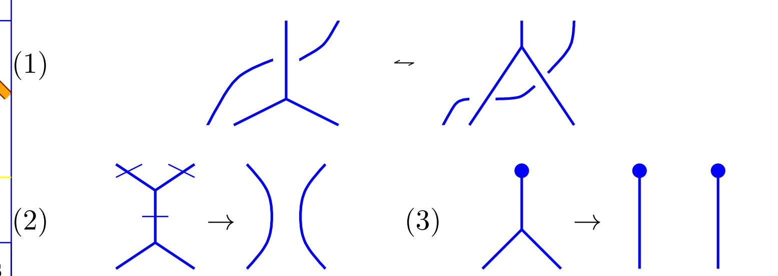


$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

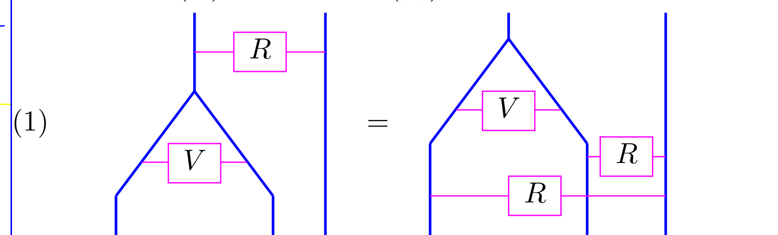
Knot-Theoretic statement.

There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

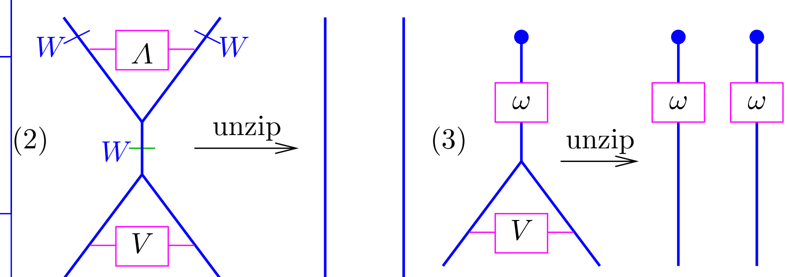


Diagrammatic statement.

Let $R = \exp \overrightarrow{\ }\in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$



(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$

(3) $V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement.

There are elements $F \in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

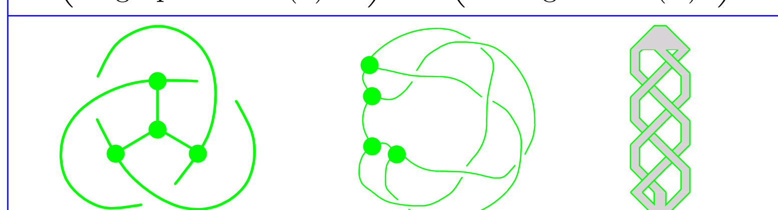
Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$, (2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and (3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$ $\Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

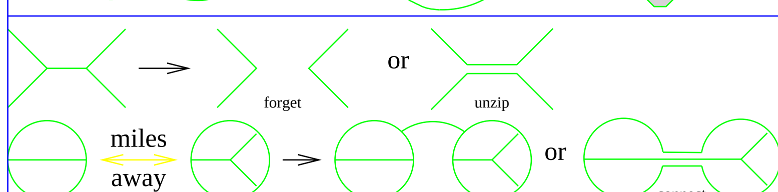
The Alekseev-Torossian Correspondence.

$$\{\text{Drinfel'd Associators}\} \leftrightarrows \{\text{Solutions of KV}\}.$$

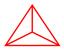
We need an even bigger algebraic structure!

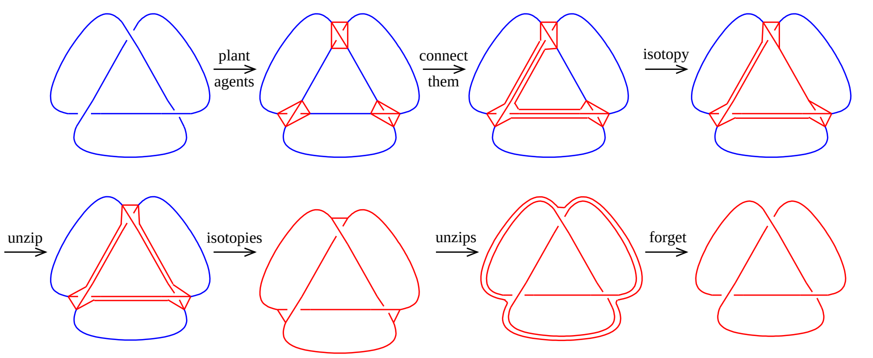
$$\left(\begin{array}{c}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{array}\right) \xrightarrow{\alpha_e} \left(\begin{array}{c}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{array}\right)$$

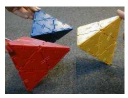


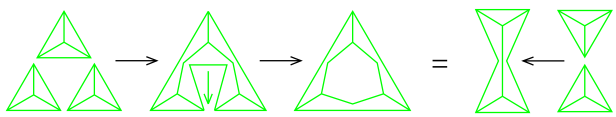


2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan, continued.

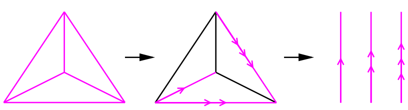
Using moves, KTG is generated by ribbon twists and the tetrahedron  : All strands here are green

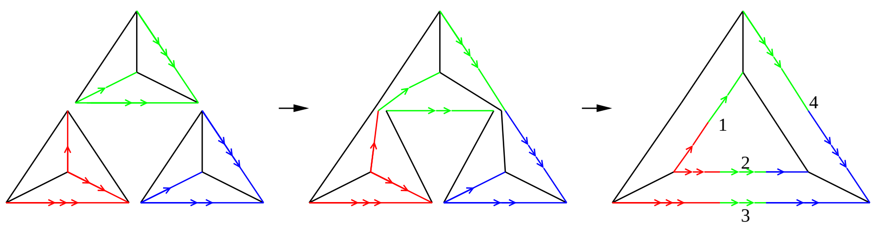


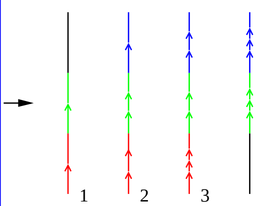
Modulo the relation(s): ( = )



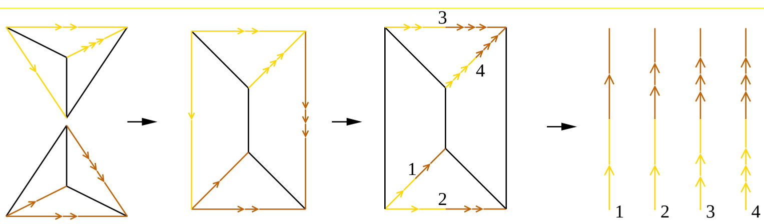
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof.  $:= \Phi \in \mathcal{A}(\uparrow_3)$



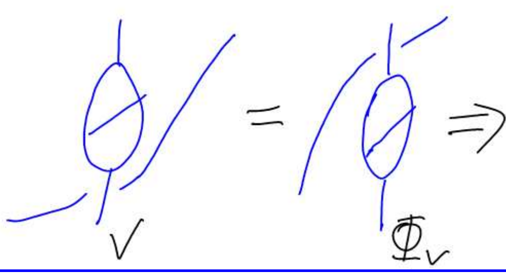


$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



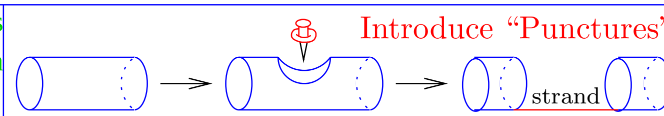
$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

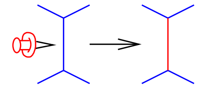
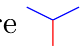
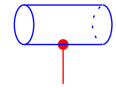
{SolKV} → {Associators}: Trivial – a tetrahedron has 4 vertices.

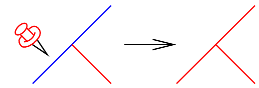


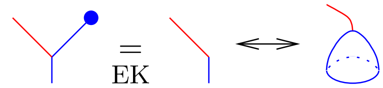
$\Phi_v \in sder_3$, so it is a sum of unrooted trees.

Introduce "Punctures".



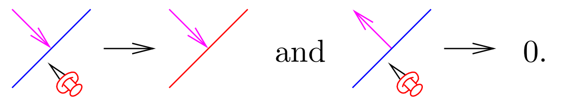
So  where  means  .

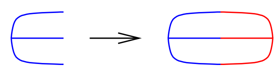
Punctures expand to the nearest **Y**-vertex: 

Note. 

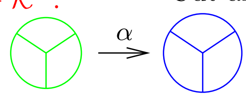
$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

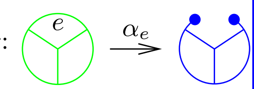
$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z: 

$\Longleftarrow$  (makes sense, and CA ops can be emulated)

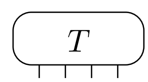
$\mathcal{K}^u \to \mathcal{K}^{\overline{w}}$. "Cut and cap is well-defined on u"

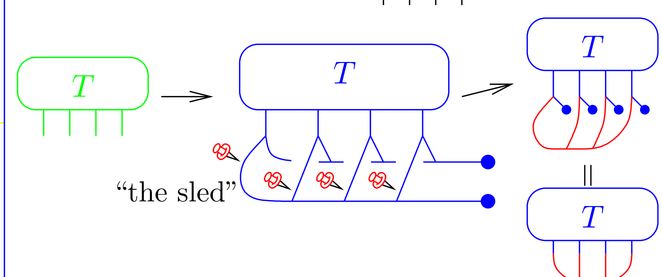
Light: 

Better: 

Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch. 

so enough to write any  . Here go:

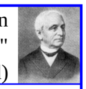


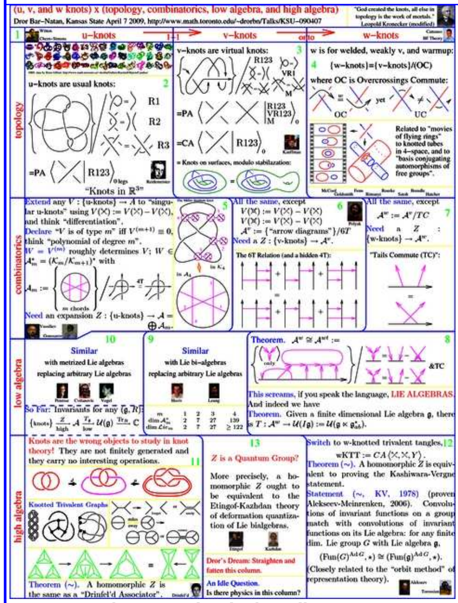
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

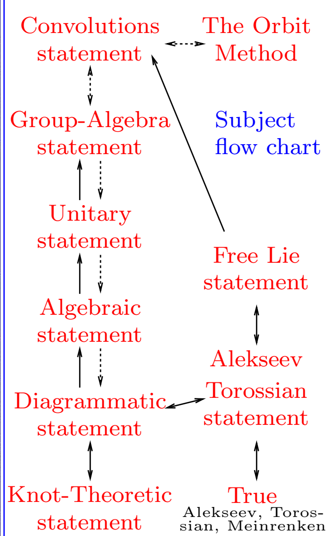
"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)





www.math.toronto.edu/~drorbn/Talks/KSU-090407

The Bigger Picture...




Kashiwara


Vergne


Alekseev


Torossian

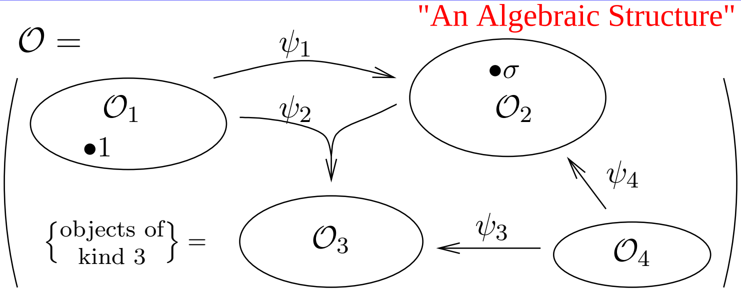
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowright \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \ldots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \ldots \end{array}$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{gr}\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

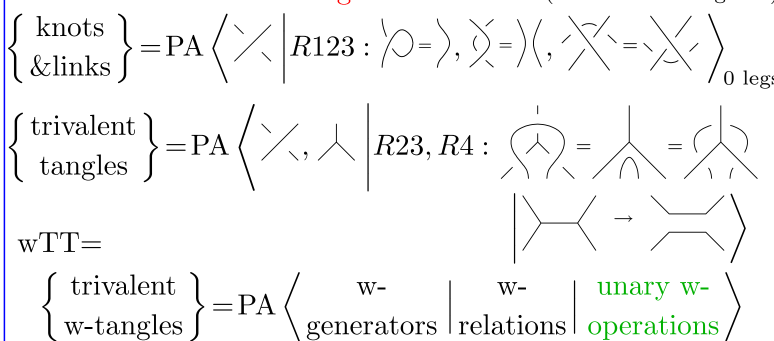
[1] http://qlink.queensu.ca/~4lb11/interesting.html

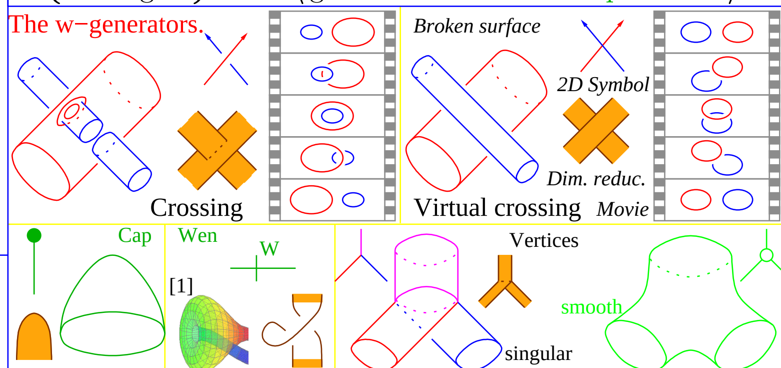
29/5/10, 8:42am

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

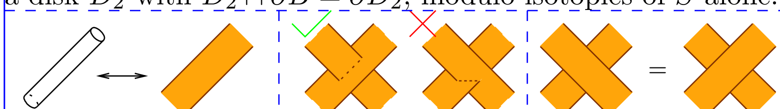
What are w-Trivalent Tangles?

(PA :=Planar Algebra)

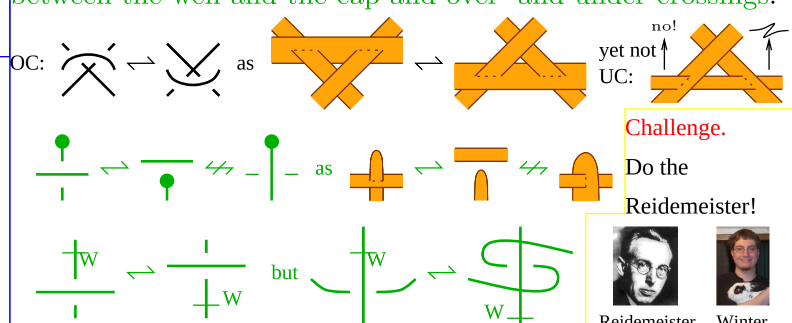


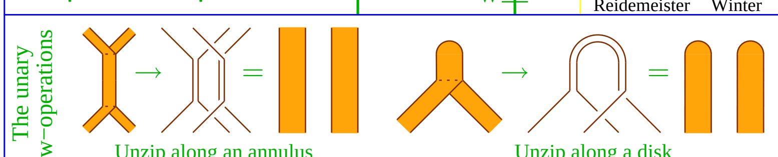


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

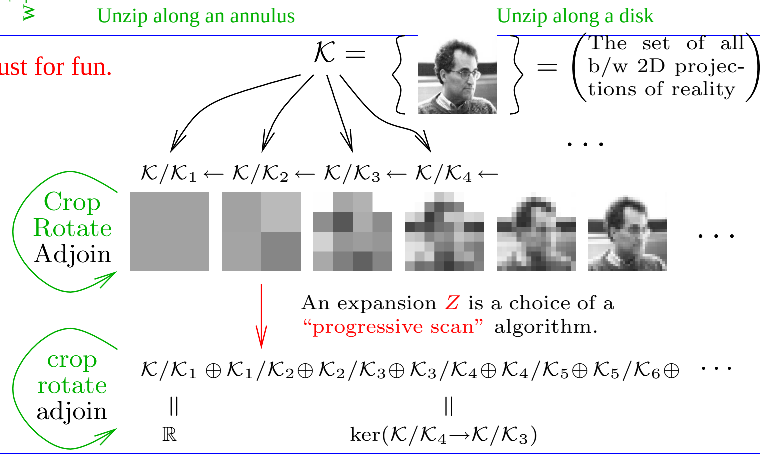


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:

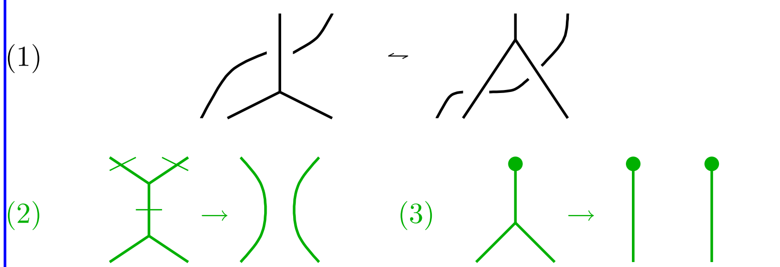




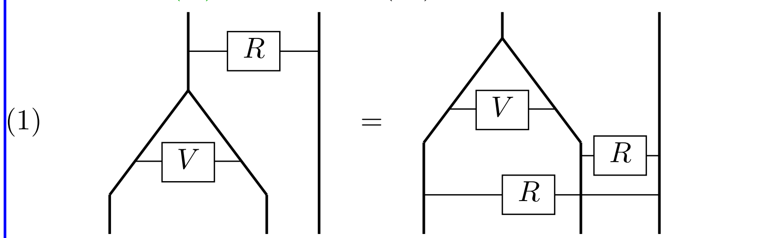
Just for fun.

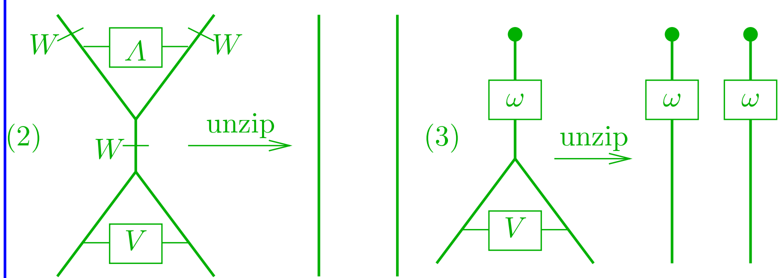


Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \mapsto \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that





Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

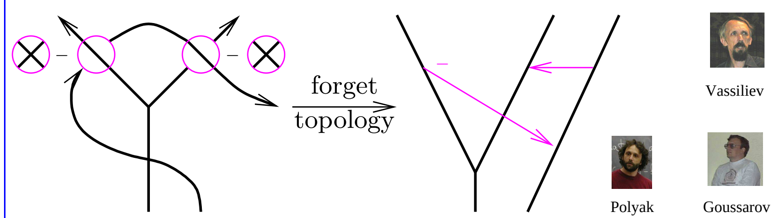
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

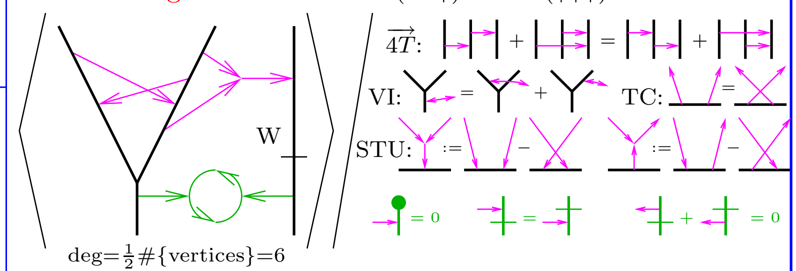
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

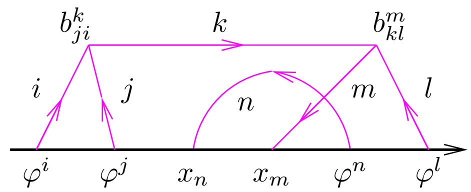
From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g},+) \ni x & \xrightarrow{\tau_0 = \exp_{\mathcal{S}}} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_{\mathcal{U}} & \downarrow \chi \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...
- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.