Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages `FreeLie`` and `AwCalculus`` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌒, ⌒}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y},\ \frac{1}{24}\overline{x\overline{\overline{xy}y}},\ -\frac{1}{720}\overline{x\overline{x\overline{x\overline{xy}}}}+\frac{1}{180}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{180}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{120}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},\ -\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1440}+\frac{1}{360}\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}+\frac{1}{240}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{1}{720}\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}-\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{1440},\ \ldots\Big]$$

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] -
      adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}}+\frac{1}{80}\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{5040}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1344}+\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{15120}+\frac{1}{840}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{3360}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{6720}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1260}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1680}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{10080},\ \ldots\Big],$$

$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}}+\frac{1}{120}\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{10080}-\frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{2016}+\frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1890}+\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{1120}+\frac{\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{5040}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{2520}+\frac{1}{840}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1260}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{5040},\ \ldots\Big]\Big\}$$

Meaningless calculations.

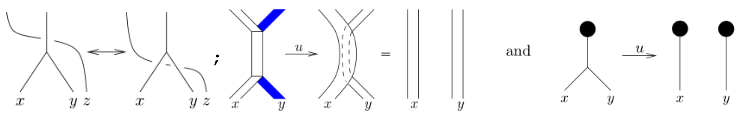
```
{b[F, G], tr_x[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{48}\overline{\overline{\overline{xy}y}y},\ \frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}-\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}-\frac{\overline{\overline{xy}\overline{\overline{xy}y}}}{1440}-\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{360}\overline{\overline{\overline{\overline{xy}y}y}y},\ \frac{\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{1440}-\frac{1}{480}\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}-\frac{1}{288}\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}-\frac{7\,\overline{x\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}}{2880}+\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{2880},\ \ldots\Big],$$

$$\mathrm{CWS}\Big[-\frac{\overline{y}}{6},\ \frac{\overline{yy}}{24},\ \frac{\overline{xxy}}{180}+\frac{\overline{xyy}}{80}-\frac{\overline{yyy}}{360},\ -\frac{\overline{xxyy}}{180}+\frac{\overline{xyxy}}{240}-\frac{\overline{xyyy}}{240}-\frac{\overline{yyyy}}{1440},\ -\frac{\overline{xxxxy}}{5040}+\frac{\overline{xxxyy}}{6720}-\frac{\overline{xxyxy}}{1120}+\frac{2\,\overline{xxyyy}}{945}-\frac{\overline{xyxyy}}{336}+\frac{\overline{xyyyy}}{6720}+\frac{\overline{yyyyy}}{10080},\ \frac{\overline{xxxxyy}}{3360}-\frac{\overline{xxxyxy}}{1344}-\frac{\overline{xxxyyy}}{2240}+\frac{\overline{xxyxxy}}{2016}+\frac{13\,\overline{xxyxyy}}{10080}+\frac{\overline{xxyyxy}}{1680}-\frac{\overline{xxyyyy}}{3780}-\frac{\overline{xyxyxy}}{840}+\frac{\overline{xyxyyy}}{5040}+\frac{\overline{xyyxyy}}{2240}+\frac{\overline{xyyyyy}}{6720}+\frac{\overline{yyyyyy}}{60480},\ \ldots\Big]\Big\}$$

(Also implemented: ∂_λ and derivations in general, tb, e^{∂_λ} and morphisms in general, div, j, Drinfel'd-Kohno, etc.)

The [BND] "vertex" equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop :
 Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

$$\Big\{\mathrm{Es}\Big[\Big\langle\overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\,\overline{x\overline{x\overline{xy}}}}{5760}-\frac{7\,\overline{x\overline{\overline{xy}y}}}{5760}+\frac{\overline{\overline{\overline{xy}y}y}}{1440},\ 0,\ -\frac{31\,\overline{x\overline{x\overline{x\overline{xy}}}}}{967680}+\frac{31\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{83\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{967680}-\frac{31\,\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{725760}-\frac{31\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{645120}+\frac{13\,\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{241920}+\frac{101\,\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{1451520}+\frac{527\,\overline{x\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}}{5806080}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{60480},\ \ldots\Big],$$

$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\overline{x\overline{xy}}}}{5760}-\frac{1}{720}\overline{x\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{\overline{xy}y}y},\ -\frac{\overline{x\overline{x\overline{x\overline{xy}}}}}{7680}+\frac{\overline{x\overline{x\overline{\overline{xy}y}}}}{3840}-\frac{\overline{\overline{x\overline{xy}}\overline{xy}}}{6912},\ -\frac{\overline{x\overline{x\overline{x\overline{x\overline{xy}}}}}}{645120}+\frac{23\,\overline{x\overline{x\overline{x\overline{\overline{xy}y}}}}}{483840}-\frac{13\,\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{161280}-\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{22680}-\frac{41\,\overline{x\overline{\overline{x\overline{xy}}\overline{xy}}}}{580608}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{15120}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{12096}+\frac{71\,\overline{x\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}}{483840}-\frac{\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}{30240},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880}+\frac{\overline{xxyy}}{2880}+\frac{\overline{xyxy}}{5760}+\frac{\overline{xyyy}}{2880},\ 0,\ -\frac{\overline{xxxxxy}}{120960}-\frac{\overline{xxxxyy}}{120960}-\frac{\overline{xxxyxy}}{120960}-\frac{\overline{xxxyyy}}{120960}-\frac{\overline{xxyxxy}}{241920}-\frac{\overline{xxyxyy}}{120960}-\frac{\overline{xxyyxy}}{120960}-\frac{\overline{xxyyyy}}{120960}-\frac{\overline{xyxyxy}}{362880}-\frac{\overline{xyxyyy}}{120960}-\frac{\overline{xyyxyy}}{241920}-\frac{\overline{xyyyyy}}{120960},\ \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11520},\ 0,\ -\frac{\overline{xxxxxx}}{725760},\ \ldots\Big]\Big\}$$

From V to F to KV following [AT].

```
logF = Λ[V]⟦1⟧ // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb]
```

```
⟨x → LS[y/2, xy/6, 1/24 xyy, -1/180 xxxy + 1/80 xxyy + 1/360 xyyy,
   -1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy -
   xyyyy/1440, xxxxxy/5040 - xxxxyy/1344 + 13 xxxyyy/15120 + 1/840 xxyxyy +
   xxxyxy/3360 + xxyyyy/6720 + xyxyyy/1260 + xxyyxy/1680 - xyyyyy/10080, …],
 y → LS[0, xy/12, 1/24 xyy, -1/360 xxxy + 1/120 xxyy + 1/180 xyyy,
   -1/720 xxxyy + 1/240 xxyyy + 1/240 xyxyy + 1/720 xxyxy -
   xyyyy/1440, xxxxxy/10080 - xxxxyy/2016 + xxxyyy/1890 + xxyxyy/1120 + xxxyxy/5040 +
   xxyyyy/2520 + 1/840 xyxyyy + xxyyxy/1260 - xyyyyy/5040, …]⟩
```

V from Z_B, following [AET, BND].

```
(El[Z_B // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη¹ // tη³ //
    hη² // hη⁴ // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])⟦
  1⟧
```

```
⟨1 → LS[0, -12/24, 0, 7 1112/5760 - 7 1122/5760 + 1222/1440, 0,
   -31 11112/967680 + 31 11122/483840 - 83 11222/967680 - 31 11212 2/725760 - 31 1 11212/645120 +
   13 1 12222/241920 + 101 12 1222/1451520 + 527 1 1212 2/5806080 - 12222 2/60480, …],
 2 → LS[1/2, -12/12, 0, 1112/5760 - 1/720 1122 + 1/720 1222,
   -11112/7680 + 11122/3840 - 11212/6912,
   -111112/645120 + 23 11122/483840 - 13 11 1222/161280 - 112 122/22680 - 41 1 11212/580608 +
   1 12222/15120 + 12 1222/12096 + 71 1 1 2212/483840 - 12222 2/30240, …]⟩
```

Solving for an associator Φ.

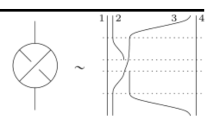
```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ -Φ) ⋀
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

```
DKS[0, 1/24 t13 t23, 0, -7 t13 t23 t23 t23/5760 + 7 t13 t13 t23 t23/5760 - t13 t13 t13 t23/1440,
 0, 31 t13 t23 t23 t23 t23 t23/967680 - 157 t13 t13 t23 t23 t13 t23/1935360 -
  31 t13 t23 t13 t23 t23 t23/387072 - 31 t13 t13 t23 t23 t23 t23/483840 +
  11 t13 t13 t13 t13 t23 t13 t23/290304 + 31 t13 t13 t23 t13 t23 t23/725760 + 83 t13 t13 t13 t23 t23 t23/967680 -
  13 t13 t13 t13 t13 t23 t23/241920 + t13 t13 t13 t13 t13 t23/60480, …]
```

The "buckle" Z_B, from Φ.



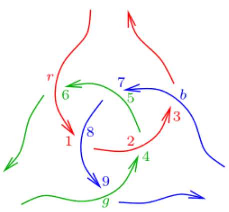
```
R = DKS[t[1, 2]/2];
Z_B = (-Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (-Φ)^σ[1,2,3] **
   Φ^σ[12,3,4];
Z_B@{4}
```

```
DKS[t23/2, -1/12 t13 t23 - 1/24 t14 t24 + 1/24 t14 t34 + 1/12 t24 t34,
 0, t13 t23 t23 t23/5760 + 7 t14 t24 t24 t24/5760 + t14 t34 t24 t24/1920 -
  t14 t34 t34 t24/1920 - 7 t14 t34 t34 t34/5760 - t24 t34 t34 t34/5760 + t14 t24 t34 t24/1920 +
  t14 t24 t14 t34/1920 - t14 t34 t24 t34/1920 - 1/720 t13 t13 t23 t23 +
  1/720 t13 t13 t13 t23 - 7 t14 t14 t24 t24/5760 + 7 t14 t14 t34 t34/5760 -
  t14 t24 t34 t34/5760 + t14 t14 t14 t24/1440 - t14 t14 t14 t34/1440 - 1/960 t14 t14 t24 t34 +
  t14 t24 t24 t34/5760 - 1/960 t24 t24 t34 t34 - t24 t24 t24 t34/5760, …]
```

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```

```
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ⟦1⟧r@{5}, ξ⟦2⟧@{5}} // Print
```



;

```
{LS[0, bg, 1/2 bbg + bgr + 1/2 bgg,
   1/6 bbbg + 1/2 bbgr + 1/2 bggr + 1/4 bbgg + 1/2 bgrr + 1/6 bggg,
   1/24 bbbbg + 1/6 bbbgr + 1/4 bbggr + 1/12 bbbgg +
   1/4 bbgrr + 1/6 bgggr + 1/4 bggrr - bbgrg +
   1/12 bbggg - 2 bbrgg + 1/6 bgrrr + 1/2 bgbgr -
   bgbrg - 1/12 bbgbg - 1/2 bgrgr + 1/24 bgggg, …],
 CWS[0, 0, 2 bgr, bbgr - bgbr + bggr - bgrg + bgrr - brgr, bbbgr/3 -
   bbgbr/2 + bbggr/2 + bbgrg/2 + bbgrr/2 + bbrbg/2 - 3 bbrgr/2 + bgbrr/2 - 3 bggbr/2 + bgggr/3 -
   bggrg/2 + bggrr/2 + bgrgg/2 - 3 bgrrg/2 + bgrrr/3 + brggr/2 - brgrr/2 + brrgr/2, …]}
```

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yet×2.

Warning. Fidgety!