PolyPoly on 150718

July-20-15 11:50 PM

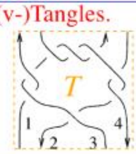
Dror Bar-Natan: Talks: Aarhus-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1507/

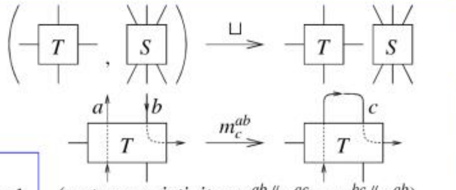
Work in Progress on

Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.

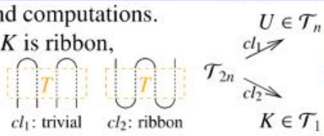




Why Tangles?

- Finitely presented. (meta-associativity: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$)
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta) \colon cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).



Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\\hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\\hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\\hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\\hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\substack{T_a, T_b \to T_c\\ \mu := 1-\beta}]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\\hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a; {}_a\!\nearrow\!\!\!\!\searrow_b, {}_b\!\nearrow\!\!\!\!\searrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\\hline a & 1\end{array}; \begin{array}{c|cc}1 & a & b\\\hline a & 1 & 1-T_a^{\pm 1}\\ b & 0 & T_a^{\pm 1}\end{array}\right).$

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)i_ :> i, ∞];
  M = Outer[Factor[∂h#1,t#2 λ] &, S, S];
  M = Prepend[M, t# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_,b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ; ∂tb,ha λ  ∂tb,hb λ  ∂tb λ; ∂ha λ  ∂hb λ  λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{hc, 1}]
    /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_,b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_,b_ := Rpa,b /. Ta → 1 / Ta;
```

Meta-Associativity Runs.

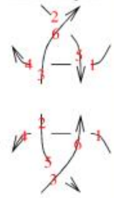
```
ξ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ξ // m12→1 // m13→1) == (ξ // m23→2 // m12→1)
```

True

R3 … divide and conquer!

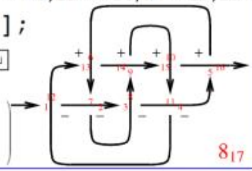
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



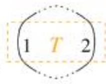
```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



Closed Components. The Halacheva trace tr_c satisfies $m^{ab}_c /\!\!/ \mathrm{tr}_c = m^{ba}_c /\!\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

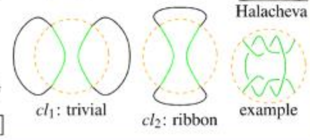
$$\begin{array}{c|cc}\omega & c & S\\\hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\\hline S & \Xi + \psi\theta/\mu\end{array}$$



Halacheva

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂tc,hc λ  ∂tc λ; ∂hc λ  λ) /. (t | h)c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ξ // m12→1 // tr1) == (ξ // m21→1 // tr1)
```

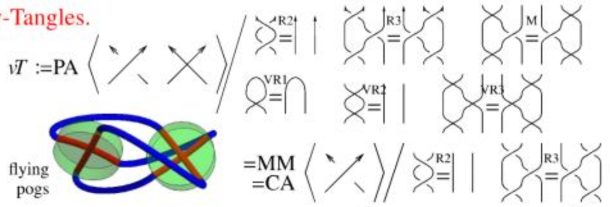
True



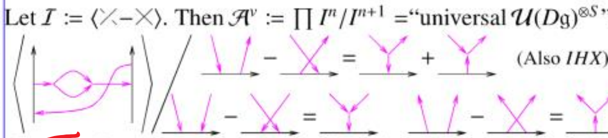
Weaknesses. • m^{ab}_c and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.



Let $\mathcal{I} := \langle \nearrow\!\!\!\!\searrow - \times \rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



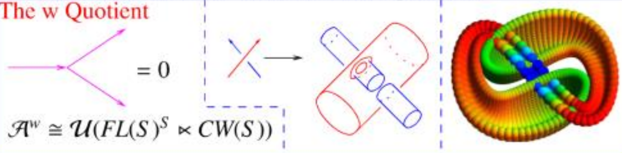
Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely

Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

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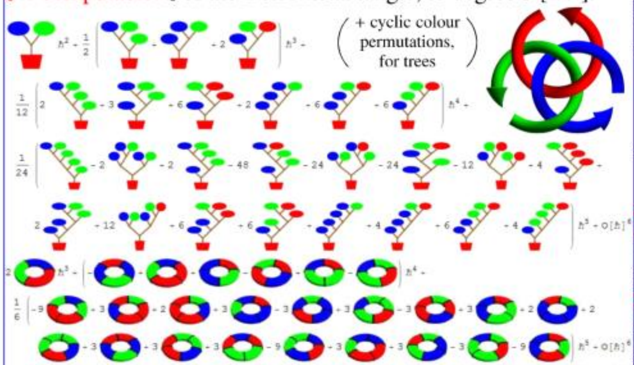
work in progress on

Polynomial Time Knot Polynomials, B

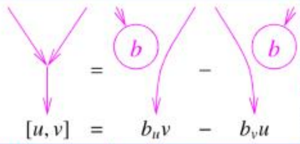
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:

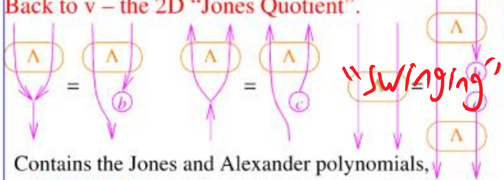


(I have a fancy free-Lie calculator!) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



Back to v – the 2D "Jones Quotient".

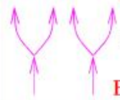


"swinging"

V. Jones

Contains the Jones and Alexander polynomials, still too hard!

The OneCo Quotient.



= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

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It should be a legal requirement that the slides of slide-based talks be linked from the conference web site *before* the actual talks.

The Abstract Context

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*$: $M(S_1)\times M(S_2)\to M(S_1\sqcup S_2)$ whenever $S_1\cap S_2=\emptyset$ and $m^{ab}_c\colon M(S)\to M((S\setminus\{a,b\})\sqcup\{c\})$ whenever $a\neq b\in S$ and $c\notin S\setminus\{a,b\}$, such that

$$\text{meta-associativity:}\quad m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$$
$$\text{meta-locality:}\quad m^{ab}_c /\!\!/ m^{de}_f = m^{de}_f /\!\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S\sqcup\{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!\!/ m^{ab}_a = Id = \epsilon_b /\!\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S\mapsto\Gamma_0(S)$ is a meta-monoid and $z_0\colon PvT\to\Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S)=\Gamma_0(S)\times\Gamma_1(S)$, along with an extension of z_0 to $z_{01}\colon PvT\to\Gamma_{01}$, with

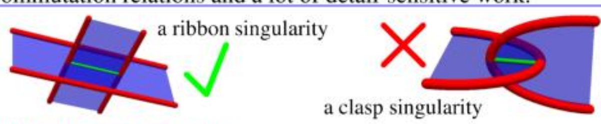
$$\Gamma_1(S) < V\oplus V^{\otimes 2}\oplus V^{\otimes 3}\oplus \mathcal{S}^2(V)^{\otimes 2}\qquad (\text{with } V:=\langle S\rangle).$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c}\colon \Gamma_1(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$, a meta-right-action $\alpha^{ab}\colon\Gamma_1(S)\times\Gamma_0(S)\to\Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c\colon\Gamma_0(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$ such that

$$(\zeta_0,\zeta_1)/\!\!/m^{ab}_c = \left(\zeta_0/\!\!/m^{ab}_{0c},\ (\zeta_1,\zeta_0)/\!\!/\alpha^{ab}/\!\!/m^{ab}_{1c} + \zeta_0/\!\!/\rho^{ab}_c\right)$$

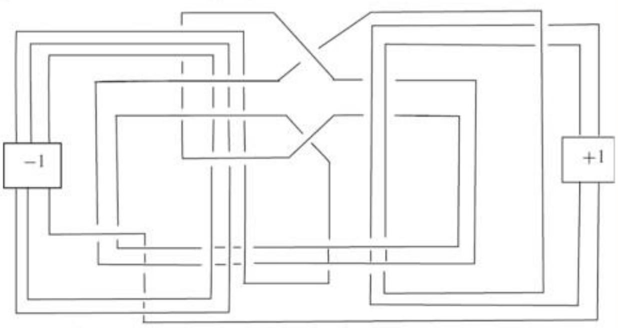
What's missing? Some commutation relations and exponentiated commutation relations and a lot of detail-sensitive work.



A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$.



[GST]: a slice knot that might not be ribbon (48 crossings).

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

Add?: It should be a legal requirement that whenever a construction is described, its computational complexity should be acknowledged.