PolyPoly on 150709

July-09-15 7:51 AM

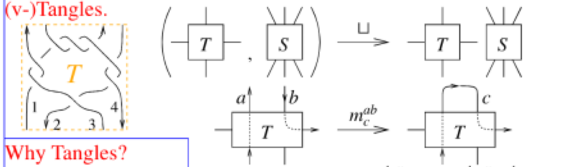
"Work in progress!"

Dror Bar-Natan: Talks: Qinhuangdao-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Qinhuangdao-1507/

Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



(meta-associativity: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$)

Why Tangles?
- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta) : cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).

cl_1: trivial cl_2: ribbon $U \in \mathcal{T}_n$, $\mathcal{T}_{2n}$, $K \in \mathcal{T}_1$

Faster is better, leaner is meaner!

Theorem 1. $\exists!$ an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\\hline S_1 & A_1\end{array},\ \begin{array}{c|c}\omega_2 & S_2\\\hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\\hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\\hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c,\ \mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\\hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\nearrow_b,\ {}_b\nwarrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\\hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\\hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

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ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
   M = Outer[Factor[∂h_a,t_b λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];

Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_,b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ε, ϕ, ψ, Ξ, μ},
   (α β θ; γ δ ε; ϕ ψ Ξ) = (∂t_a,h_a λ  ∂t_a,h_b λ  ∂t_a λ; ∂t_b,h_a λ  ∂t_b,h_b λ  ∂t_b λ; ∂h_a λ  ∂h_b λ  λ) /. (t | h)a|b → 0;
   Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ  ε + δ θ / μ; ϕ + α ψ / μ  Ξ + ψ θ / μ).{hc, 1}]
     /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_,b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_,b_ := Rpab /. Ta → 1 / Ta;
```

Meta-Associativity Runs.

$\zeta = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_s\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_s\}\right];$

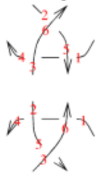
$(\zeta\ //\ m_{12\to1}\ //\ m_{13\to1}) == (\zeta\ //\ m_{23\to2}\ //\ m_{12\to1})$

True

R3 … divide and conquer!

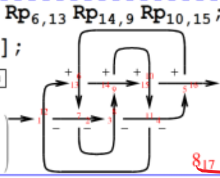
$\{Rm_{51}\ Rm_{62}\ Rp_{34}\ //\ m_{14\to1}\ //\ m_{25\to2}\ //\ m_{36\to3},$
$Rp_{61}\ Rm_{24}\ Rm_{35}\ //\ m_{14\to1}\ //\ m_{25\to2}\ //\ m_{36\to3}\}$

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix},\ \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



$z = Rm_{12,1}\ Rm_{27}\ Rm_{83}\ Rm_{4,11}\ Rp_{16,5}\ Rp_{6,13}\ Rp_{14,9}\ Rp_{10,15};$
$Do[z = z\ //\ m_{1k\to1},\ \{k, 2, 16\}];$
z

$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}

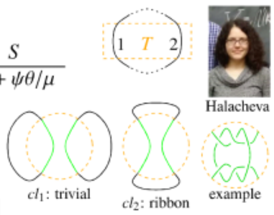
"Pixelation issues."

Closed Components. The Halacheva trace satisfies $m^{ab}_c /\!\!/ \mathrm{tr}_c = m^{ba}_c /\!\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\\hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\\hline S & \Xi+\psi\theta/\mu\end{array}$$

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
   (α θ; ψ Ξ) = (∂t_c,h_c λ  ∂t_c λ; ∂h_c λ  λ) /. (t | h)c → 0;
   Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

True

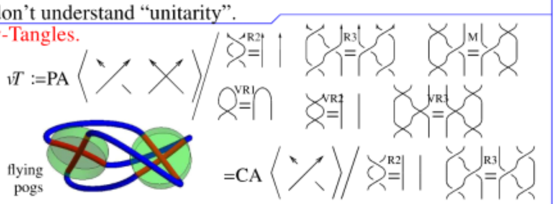


cl_1: trivial cl_2: ribbon example Halacheva

Weaknesses. • m^{ab}_c and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c. • I still don't understand "unitarity".

"breeze through (~~grey out?~~)"

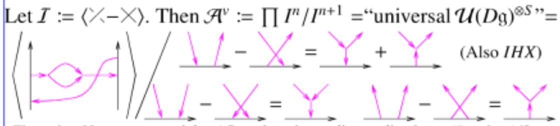
v-Tangles.



$v\mathcal{T}$:=PA ⟨ ⟩ / R2, R3, M, VR1, VR2, VR3

flying pogs =CA ⟨ ⟩ / R2, R3

Let $\mathcal{I} := \langle \asymp - \times \rangle$. Then $\mathcal{A}^v := \prod \mathcal{I}^n/\mathcal{I}^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



(Also IHX)

Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$.

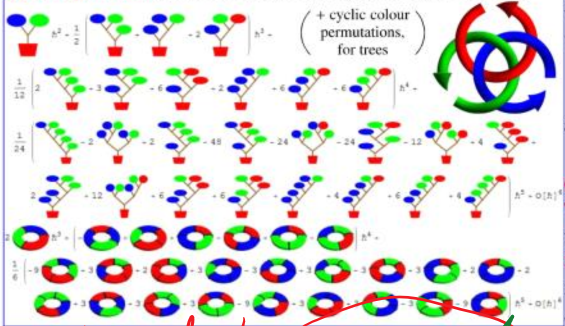
Too hard! Let's look for "meta-monoid" quotients.

Something about w

Polynomial Time Knot Polynomials, B

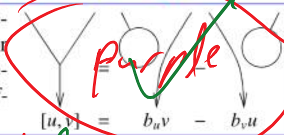
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable! z of the Borromean tangle, to degree 5 [BN]:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.

Put below Jones



The "Jones" quotient

$\boxed{\text{Jones}}$

The 1-co-quotient

everything should be polynomial...

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S \sqcup \{a, b\}) \to M(S \sqcup \{c\})$ whenever $a \neq b \notin S$ and $c \notin S$, such that

The Abstract Context

$$\text{meta-associativity:} \quad m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$$

$$\text{meta-locality:} \quad m^{ab}_c /\!\!/ m^{de}_f = m^{de}_f /\!\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:} \quad \epsilon_b /\!\!/ m^{ab}_a = Id = \epsilon_b /\!\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : \mathit{PvT} \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : \mathit{PvT} \to \Gamma_{01}$, with (more or less)

$$\Gamma_1(S) =$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and an order 1 differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

...

and then

$$(\zeta_0, \zeta_1) /\!\!/ m^{ab}_c = \left(\zeta_0 /\!\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!\!/ \alpha^{ab} /\!\!/ m^{ab}_{1c} + \zeta_0 /\!\!/ \rho^{ab}_c \right)$$

References.

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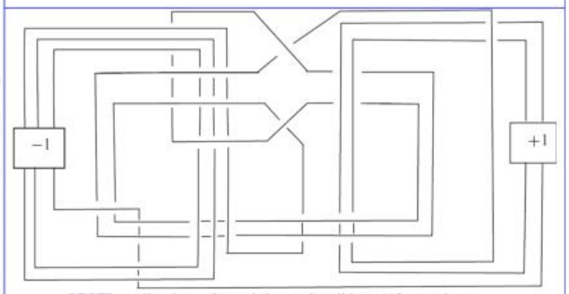

[GST]: a slice knot that might not be ribbon (48 crossings).

Let's talk about China, America, Taiwan, economy, ecology, religion, democracy, censorship, and all else.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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add: Something about one-co, abc calculus,

"I'm 2D and I'm a friend of Vaughan Jones"