PolyPoly on 150707

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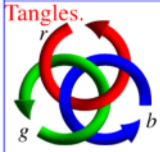
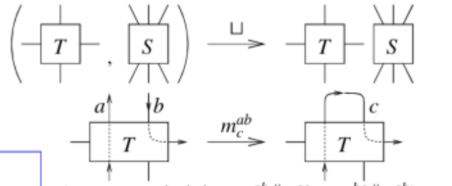
Dror Bar-Natan: Talks: Qinhuangdao-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Qinhuangdao-1507/

Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

Tangles.





(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?
- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $Z(K) \in \{cl_2(Z) \colon cl_1(Z) = 1\}$.
- (Genus and crossing number are also definable properties).

Faster is better, leaner is meaner!

Theorem. $\exists$ an invariant $Z\colon$ {pure framed S-component tangles} $\to R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

1. $\left(\begin{array}{c|c} \omega_1 & S_1 \\ \hline S_1 & A_1 \end{array}, \begin{array}{c|c} \omega_2 & S_2 \\ \hline S_2 & A_2 \end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$,

2. $\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc} \mu\omega & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array}\right)_{T_a,T_b\to T_c}$,

and satisfying $\left(\uparrow_a;\ {}_a\!\!\nearrow\!\!\nwarrow_b,\ {}_b\!\!\nwarrow\!\!\nearrow_a\right) \xrightarrow{Z} \left(\begin{array}{c|c} 1 & a \\ \hline a & 1 \end{array};\ \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm1} \\ b & 0 & T_a^{\pm1} \end{array}\right)$.

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
   M = Outer[Factor[∂h_α1 t_α2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ  ϵ + δ θ / μ; ϕ + α ψ / μ  Ξ + ψ θ / μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1 - T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;
```

Meta-Associativity **Runs.**

```
η = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(η // m_{12→1} // m_{13→1}) == (η // m_{23→2} // m_{12→1})
```

True

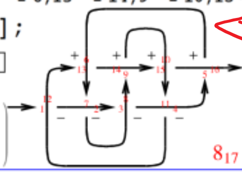
R3 … divide and conquer!

```
{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
 Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$

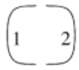

```
Z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[Z = Z // m_{1k→1}, {k, 2, 16}];
Z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$



Closed Components. The Halacheva trace satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

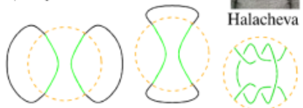
$\begin{array}{c|cc} \omega & c & S \\ \hline c & \alpha & \theta \\ S & \psi & \Xi \end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c} \mu\omega & S \\ \hline S & \Xi+\psi\theta/\mu \end{array}$



Halacheva

```
tr_{c_}[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c}λ ∂_{t_c}λ; ∂_{h_c}λ λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(η // m_{12→1} // tr_1) == (η // m_{21→1} // tr_1)
```

True



Weaknesses. • I still don't understand "unitarity".
- m^{ab}_c, tr_c are non-linear.
- The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions.

A v-view comes here!

Add 1. Some tr_c computations? The Borromean?
2. The relationship w/ A^w w/ an explicit map!

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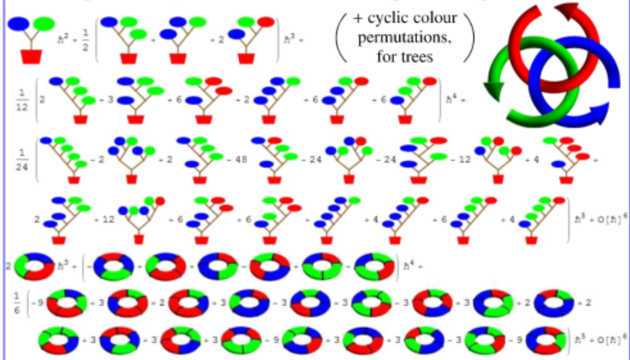
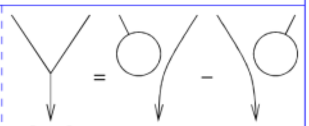
Polynomial Time Knot Polynomials, B

Theorem 3 [BND, BN]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.



$$[u, v] = c_u v - c_v u$$

References.

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[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

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[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Let's talk about China, America, Taiwan, economy, ecology, religion, democracy, censorship, and all else.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Really add:

Something about one-co, abc calculus,

"I'm 2D and I'm a friend of Vaughan Jones"

The Abstract Context.

Definition. A meta monoid is a functor

M: (Finite sets, injections) $\longrightarrow$ Sets

(think "$M(S)$ is

quantum G^S", for G a group) along

with natural operations

$*: M(S_1)\times M(S_2)\longrightarrow M(S_1\sqcup S_2)$ whenever $S_1\cap S_2=\emptyset$

$*$: $m^{ab}_c: M(S\cup\{a,b\})\to M(S\cup\{c\})$

whenever $a\neq b\notin S$, $c\notin S$

Such that:

1. meta-associativity:

$$m^{ab}_a // m^{ac}_a = m^{bc}_b // m^{ab}_a$$

2. meta-unit: with $\epsilon_b: M(S\to S\cup\{b\})$,

$$\epsilon_b // m^{ab}_a = Id = \epsilon_b // m^{ba}_a$$

Claim. Virtual tangles for a meta monoid.

Thm. $S \to \Gamma_0(S)$ is a meta-monoid, and ζ_0 is a morphism of meta-monoids.

Strong conviction. There exists an extention of Γ_0 to a bigger MM,

$$\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S),$$

along with an extention of ζ_0 to

$$\zeta_{01} : vT \longrightarrow \Gamma_{01}$$

with (more or less),

$$\Gamma_1(S) = R$$

Furthermore, upon reducing to a single variable, everything is polynomial size and polynomial time.

P.S. Γ_1 is a "meta 2-cocycle over Γ_0":

There is $m_{1c}^{ab} : \Gamma_1(S \cup \{a,b\}) \longrightarrow \Gamma_1(S \cup \{c\})$
and $\rho_c^{ab} : \Gamma_0(S \cup \{a,b\}) \longrightarrow \Gamma_1(S \cup \{c\})$

such that ...

and then

$$m_c^{ab}(\zeta_0, \zeta_1) = \left(m_{0c}^{ab}(\zeta_0), \rho_c^{ab}(\zeta_0) + m_{1c}^{ab}(\zeta_1)\right)$$