PolyPoly on 150707-2

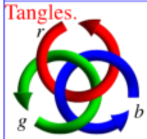
July-07-15 2:15 AM

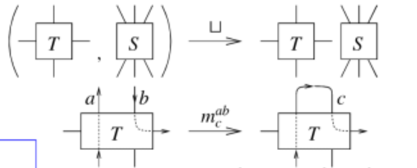
Dror Bar-Natan: Talks: Qinhuangdao-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Qinhuangdao-1507/

Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

Tangles.

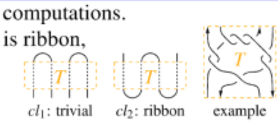




(meta-associativity: $m_a^{ab} /\!\!/ m_a^{ac} = m_b^{bc} /\!\!/ m_a^{ab}$)

Why Tangles?
- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon,
  $z(K) \in \{cl_2(\zeta) : cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).



Faster is better, leaner is meaner!

Theorem 1. $\exists!$ an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

1. $$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array},\ \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

2. $$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right)_{T_a,T_b\to T_c},$$

and satisfying $\left(\uparrow_a;\ {}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea: ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
  M = Outer[Factor[∂h_a1,t_a2 λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_a_,b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂t_a,h_a λ  ∂t_a,h_b λ  ∂t_a λ; ∂t_b,h_a λ  ∂t_b,h_b λ  ∂t_b λ; ∂h_a λ  ∂h_b λ  λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ/μ  ϵ + δ θ/μ; ϕ + α ψ/μ  Ξ + ψ θ/μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_a_,b_ := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_a_,b_ := Rp_ab /. T_a → 1/T_a;
```

Meta-Associativity Runs.

```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```
True

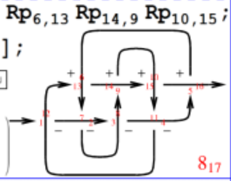
R3 … divide and conquer!

```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix},\ \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$


```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

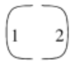
$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



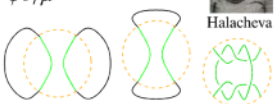
Closed Components. The Halacheva trace satisfies $m_c^{ab} /\!\!/ \mathrm{tr}_c = m_c^{ba} /\!\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi+\psi\theta/\mu\end{array}$$

```
tr_c_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂t_c,h_c λ  ∂t_c λ; ∂h_c λ  λ) /. (t | h)c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```
True



Halacheva



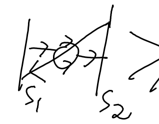
I still don't understand trc

Weaknesses. • I still don't understand "unitarity".
- $m_c^{ab} /\!\!/ \mathrm{tr}_c$ are non-linear.
- The product ωA is always Laurent, but my current proof takes and induction with exponentially many conditions.

$vT = PA(\curlyvee, \times / \text{rels})$
$= CA(\curlyvee)/\text{rels}$ ~~A v-view comes here!~~

Dog image

Let $I = \langle \curlyvee - \times \rangle$. Then
$A^v = gr_I vT = \bigoplus I^n / I^{n+1} =$

$\langle$  $\rangle / STU_1, STU_2, STU_3, IHX$

$= \text{Combinatorial } U(\mathfrak{g} \oplus \mathfrak{g}^*)^{\otimes S}$

<u>Thm</u> (Etingof-Kazhdan, Enriques)
There exists a homomorphic expansion
$Z: vT \to A^v$

Add! Some trc computations? The Borromean?

2. The relationship w/ A^w w/ an explicit map!

compare [BNS, BN]

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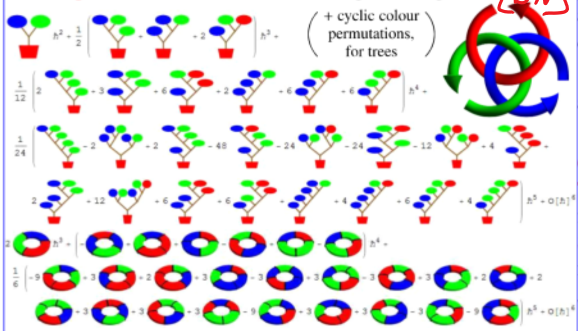
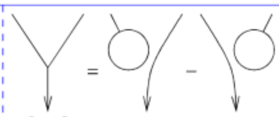
Polynomial Time Knot Polynomials, B

Theorem 2 [BND, ~~BN~~]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of ~~w-knotted balloons and hoops~~ tangles. $z^w := \log Z^w$ takes values in $FL(\cancel{T}\,S)^T \times CW(\cancel{T}\,S)$.

z is computable! z of the Borromean tangle, to degree 5: [BN]



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to ~~z_0~~ z_0 ~~and the KBH operations on z reduce to the formulas in Theorem 1~~.



$$[u,v] = c_u v - c_v u$$

The Abstract Context

Definition. A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1)\times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S \sqcup \{a,b\}) \to M(S \sqcup \{c\})$ whenever $a \neq b \notin S$ and $c \notin S$, such that

$$\text{meta-associativity:}\quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

independence: $m^{ab} m^{cd} = m^{cd} m^{ab}$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a .$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with (more or less)

$$\Gamma_1(S) = $$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, $\cancel{\Gamma}$ Γ_{01} is given using a "meta-2-cocycle over Γ_0": there are $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$ and $\rho^{ab}_c : \Gamma_0(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

...

and then

$$m^{ab}_c(\zeta_0, \zeta_1) = \left(m^{ab}_{0c}(\zeta_0), m^{ab}_{1c}(\zeta_a) + \rho^{ab}_c(\zeta_1).\right)$$

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

add EK, ~~E~~

Let's talk about China, America, Taiwan, economy, ecology, religion, democracy, censorship, and all else.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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Really add:

Something about one-co, abc calculus,

"I'm 2D and I'm a friend of Vaughan Jones"

Furthermore, Γ_1 is given using a "meta-2-cocycle over Γ_0": there are $m^{ab}_{1c}\colon \Gamma_1(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$ and $\rho^{ab}_c\colon \Gamma_0(S\sqcup\{a,b\})\to\Gamma_1(S\sqcup\{c\})$ such that

...

right Action $\alpha^{ab}\colon \Gamma_1(S)\times\Gamma_0(S)\to\Gamma_1(S)$ linear in 1st variable.

$m^{ab}_{1c}\colon \Gamma_1(S\cup\{a,b\})\to\Gamma_1(S\cup\{c\})$

linear?

R-linear?

$$(\zeta_0,\zeta_1)/\!/m^{ab}_c = \left(\zeta_0/\!/m^{ab}_{0c},\ (\zeta_1,\zeta_0)/\!/\alpha^{ab}/\!/m^{ab}_{1c} + \zeta_0/\!/\rho^{ab}_c\right)$$ TODO?