PolyPoly in Qinhuangdao, Post Mortem

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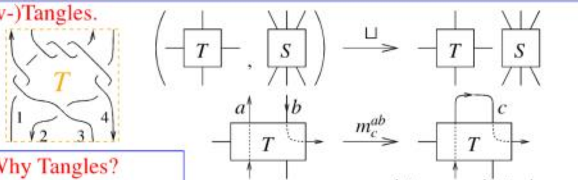
Dror Bar-Natan: Talks: Qinhuangdao-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Qinhuangdao-1507/

Work in Progress on **Polynomial Time Knot Polynomials, A**

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta)\colon cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).

$U \in \mathcal{T}_n$, $\mathcal{T}_{2n}$, $K \in \mathcal{T}_1$; cl_1: trivial, cl_2: ribbon

Faster is better, leaner is meaner!

Theorem 1. ∃! an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c,\ \mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\!\times_b,\ {}_b\!\times_a\right) \mapsto \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm 1}\\ b & 0 & T_a^{\pm 1}\end{array}\right).$

Font

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].

- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A - I)/(1 - T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

Implementation key idea:

$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

ωεβ/Demo

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2]
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ε, φ, ψ, Ξ, μ},
  (α β θ; γ δ ε; φ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t|h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + αδ/μ  ε + δθ/μ; φ + αψ/μ  Ξ + ψθ/μ).{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // ΓCollect];
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_{a_} :→ a, ∞];
  M = Outer[Factor[∂_{t_#1 h_#2}λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1 1-T_a; 0 T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1/T_a;
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Meta-Associativity — Runs.

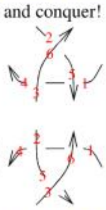
```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; φ1 φ2 φ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```

True

R3 ... divide and conquer!

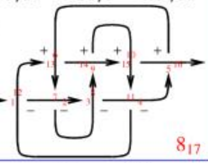
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}

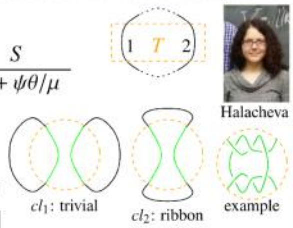
Closed Components. The Halacheva trace satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi + \psi\theta/\mu\end{array}$$

Font

```
tr_c_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c}λ ∂_{t_c}λ; ∂_{h_c}λ λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

True



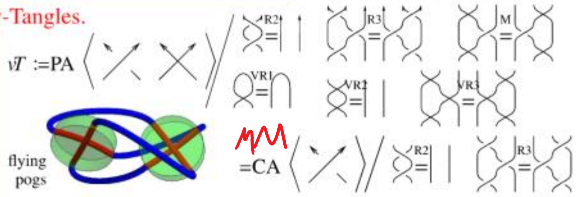
Halacheva

cl_1: trivial cl_2: ribbon example

Weaknesses. • m^{ab}_c and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c. • I still don't understand "unitarity".

Where does it come from?

v-Tangles.

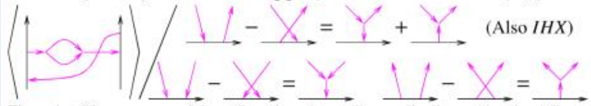


$v\mathcal{T}$:=PA ⟨ crossings ⟩ / R2, R3, M, VR1, VR2, VR3 ; =CA ⟨ ⟩ / R2, R3

MM; In Analogy

flying pogs

Let $I := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ ="universal $\mathcal{U}(\mathcal{D}g)^{\otimes S}$"=



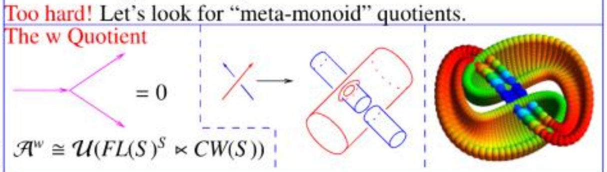
(Also IHX)

Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$.

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient



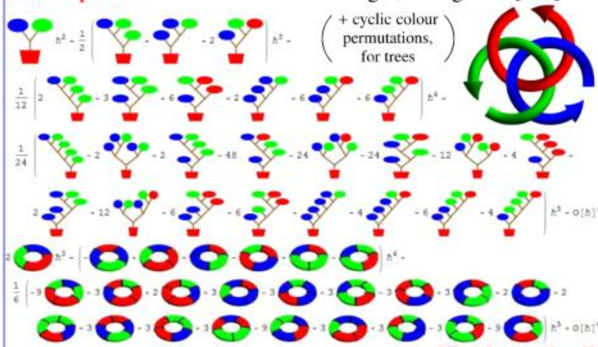
$= 0$

$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

Polynomial Time Knot Polynomials, B

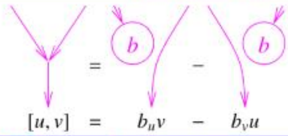
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:

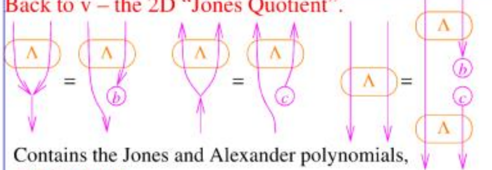


(I have a fancy free-Lie calculator!) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



Back to v – the 2D "Jones Quotient".



V. Jones

Contains the Jones and Alexander polynomials, still too hard!

The OneCo Quotient.

= 0, only one co-bracket is allowed.

Everything should work, and everything is being worked!

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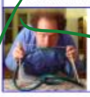


Let's talk about China, America, Taiwan, economy, ecology, religion, democracy, censorship, and all else.

in Aarhus.

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c: M(S) \to M((S \setminus \{a,b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a,b\}$, such that

The Abstract Context

$$\text{meta-associativity:}\quad m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$$

$$\text{meta-locality:}\quad m^{ab}_c /\!\!/ m^{de}_f = m^{de}_f /\!\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:}\quad \epsilon_b /\!\!/ m^{ab}_a = Id = \epsilon_b /\!\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0: PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01}: PvT \to \Gamma_{01}$, with

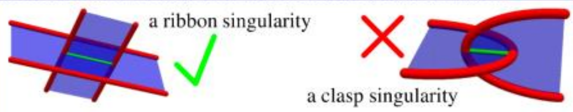
$$\Gamma_1(S) < \langle S \times S \times S \sqcup S \times S \times S \times S \rangle.$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c}: \Gamma_1(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab}: \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c: \Gamma_0(S \sqcup \{a,b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

$$(\zeta_0, \zeta_1) /\!\!/ m^{ab}_c = \left(\zeta_0 /\!\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!\!/ \alpha^{ab} /\!\!/ m^{ab}_{1c} + \zeta_0 /\!\!/ \rho^{ab}_c\right)$$

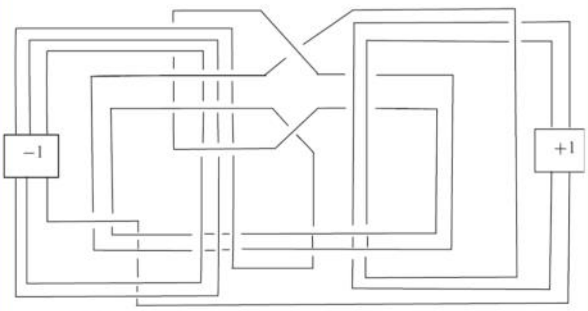
What's missing? Some commutation relations and exponentiated commutation relations and a lot of detail-sensitive work.



A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$.



[GST]: a slice knot that might not be ribbon (48 crossings).

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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