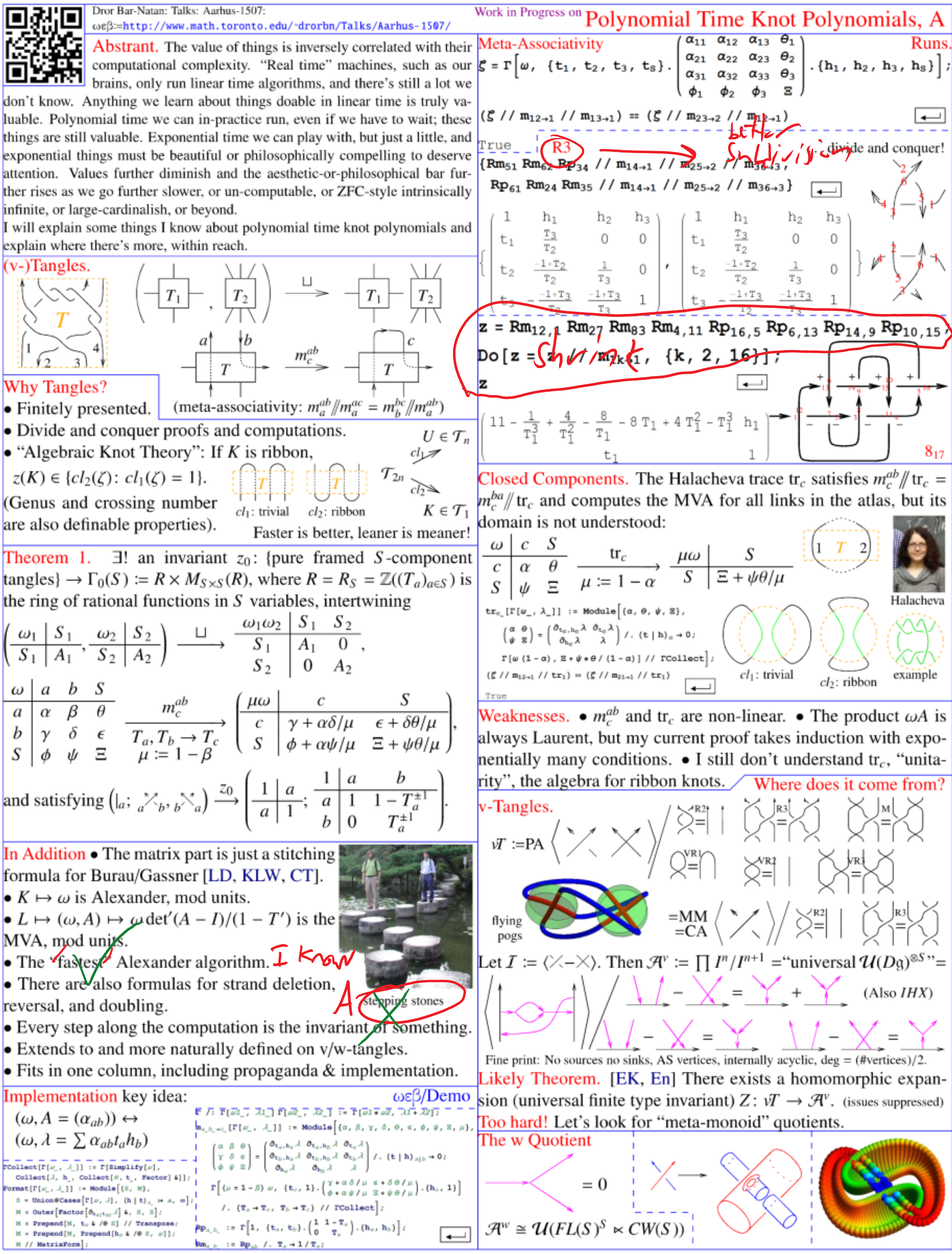
PolyPoly in Aarhus, Post Mortem

July-31-15 8:58 AM

I failed to emphasize that Γ-calculus leads to a polynomial time algorithm.



Dror Bar-Natan: Talks: Aarhus-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1507/

Work in Progress on **Polynomial Time Knot Polynomials, A**

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.
I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles. (meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?
- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta) : cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties). cl_1: trivial, cl_2: ribbon. Faster is better, leaner is meaner!

Theorem 1. $\exists!$ an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\\hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\\hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\\hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\\hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a,T_b\to T_c,\ \mu:=1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\\hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $(\uparrow_a;\ {}_a\nearrow_b, {}_b\nwarrow_a) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\\hline a & 1\end{array}; \begin{array}{c|cc}1 & a & b\\\hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right)$.

In Addition • The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega\det'(A-I)/(1-T')$ is the MVA, mod units.
- The "fastest" Alexander algorithm. I know
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

A stepping stones

Implementation key idea: ωεβ/Demo
$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

Meta-Associativity **Runs.**

$\zeta = \Gamma[\omega, \{t_1, t_2, t_3, t_S\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_S\}]$;

(ζ // m12→1 // m13→1) == (ζ // m23→2 // m21→1)

True — R3 — better subdivision — divide and conquer!

{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3, Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}

z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15 ;
Do[z = ~~z // m~~ shrink k1, {k, 2, 16}]; z

8_{17}

Closed Components. The Halacheva trace tr_c satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$\begin{array}{c|cc}\omega & c & S\\\hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\\hline S & \Xi+\psi\theta/\mu\end{array}$

Halacheva — cl_1: trivial — cl_2: ribbon — example

Weaknesses. • m^{ab}_c and tr_c are non-linear. • The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions. • I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles. $v\mathcal{T}$:= PA ⟨ ... ⟩ — flying pogs — =MM =CA

Let $I := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ ="universal $\mathcal{U}(\mathcal{D}\mathfrak{g})^{\otimes S}$"= (Also IHX)

Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem. [EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z: v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient = 0

$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

A: Polyank & Ohtsuki @Heian Shrine, Kyoto

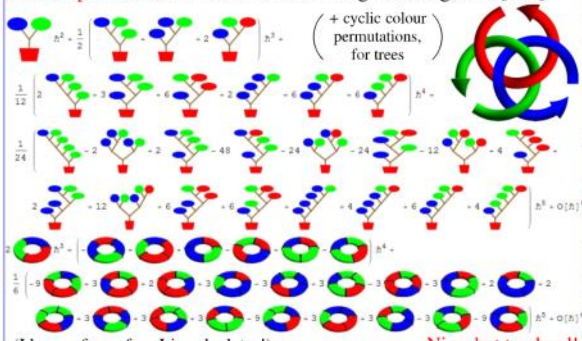
B: Add some DuField stats?

Dror Bar-Natan: Talks: Aarhus-1507:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1507/

Work in Progress on **Polynomial Time Knot Polynomials, B**

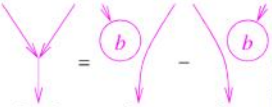
Theorem 2 [BND]. ∃! a homomorphic expansion, aka a homomorphic universal finite type invariant Z^w of pure w-tangles. $z^w := \log Z^w$ takes values in $FL(S)^S \times CW(S)$.

z is computable. z of the Borromean tangle, to degree 5 [BN]:



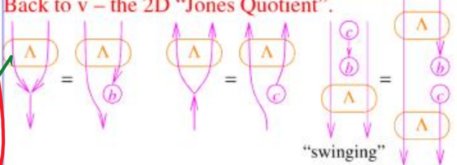
(I have a fancy free-Lie calculator!) Nice, but too hard!

Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, z^w reduces to z_0.



Compress

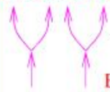
Back to v – the 2D "Jones Quotient".



V. Jones

Contains the Jones and Alexander polynomials, . . . still too hard!

The OneCo Quotient.



= 0, only one co-bracket is allowed. E.

Everything should work, and everything is being worked!

Definition. (Compare [BNS, BN]) A meta-monoid is a functor M: (finite sets, injections)→(sets) (think "$M(S)$ is quantum G^S", for G a group) along with natural operations $*: M(S_1) \times M(S_2) \to M(S_1 \sqcup S_2)$ whenever $S_1 \cap S_2 = \emptyset$ and $m^{ab}_c : M(S) \to M((S \setminus \{a, b\}) \sqcup \{c\})$ whenever $a \neq b \in S$ and $c \notin S \setminus \{a, b\}$, such that

The Abstract Context

$$\text{meta-associativity:} \quad m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$$

$$\text{meta-locality:} \quad m^{ab}_c /\!/ m^{de}_f = m^{de}_f /\!/ m^{ab}_c$$

and, with $\epsilon_b = M(S \hookrightarrow S \sqcup \{b\})$,

$$\text{meta-unit:} \quad \epsilon_b /\!/ m^{ab}_a = Id = \epsilon_b /\!/ m^{ba}_a.$$

Claim. Pure virtual tangles PvT form a meta-monoid.

Theorem. $S \mapsto \Gamma_0(S)$ is a meta-monoid and $z_0 : PvT \to \Gamma_0$ is a morphism of meta-monoids.

Strong Conviction. There exists an extension of Γ_0 to a bigger meta-monoid $\Gamma_{01}(S) = \Gamma_0(S) \times \Gamma_1(S)$, along with an extension of z_0 to $z_{01} : PvT \to \Gamma_{01}$, with

$$\Gamma_1(S) < V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \mathcal{S}^2(V)^{\otimes 2} \qquad (\text{with } V := \langle S \rangle).$$

Furthermore, upon reducing to a single variable everything is polynomial size and polynomial time.

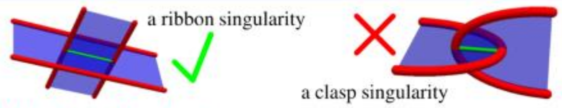
Furthermore, Γ_{01} is given using a "meta-2-cocycle ρ^{ab}_c over Γ_0": In addition to $m^{ab}_c \to m^{ab}_{0c}$, there are R_S-linear $m^{ab}_{1c} : \Gamma_1(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$, a meta-right-action $\alpha^{ab} : \Gamma_1(S) \times \Gamma_0(S) \to \Gamma_1(S)$ R_S-linear in the first variable, and a first order differential operator (over R_S) $\rho^{ab}_c : \Gamma_0(S \sqcup \{a, b\}) \to \Gamma_1(S \sqcup \{c\})$ such that

Snakes ~~ladders~~
Print[VerifyR3[#]] &

$$(\zeta_0, \zeta_1) /\!/ m^{ab}_c = \left(\zeta_0 /\!/ m^{ab}_{0c}, (\zeta_1, \zeta_0) /\!/ \alpha^{ab} /\!/ m^{ab}_{1c} + \zeta_0 /\!/ \rho^{ab}_c \right)$$

What's missing? Some commutation relations and exponentiated commutation relations and a lot of detail-sensitive work.

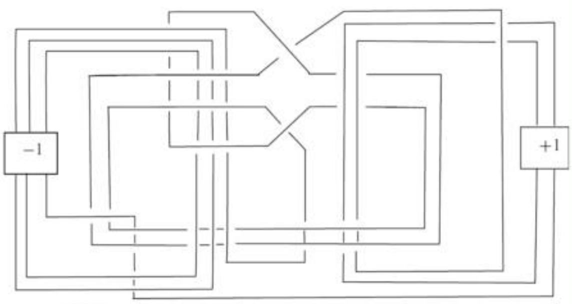
add ρ_2



A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon. slice

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$.



[GST]: a slice knot that might not be ribbon (48 crossings).

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430–479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Help Needed!

I'm slow and feeble-minded.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

E: Likely related to [ADO].

Also include refs to the Lawrence representations and to Ito's paper(s).