Pensieve header: One-Co computations in the abc presentation; continued pensieve://2015-08/.

The bracket

On the elements β, a, c, δa, ca, δaa.

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
OQ[is___] := OrderedQ[{is}];

Simp[expr_] := Simplify[expr];
S[β[f_]] := β[Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[c[f_, k_]] := c[Simp[f], k];
S[δa[f_, i_, j_]] := δa[Simp[f], i, j];
S[ca[f_, j_, k_, l_]] := ca[Simp[f], j, k, l];
S[δaa[f_, i_, j_, k_, l_]] := δaa[Simp[f], i, j, k, l];
S[expr_] := expr /. (λ_β | λ_a | λ_δa | λ_c | λ_ca | λ_δaa) ⧴ S[λ];

β[0] := 0;
β /: β[f_] + β[g_] := β[f + g] // S;
β /: g_ * β[f_] := β[g f] // S;
a[0, _, _] := 0;
a /: a[f_, j_, k_] + a[g_, j_, k_] := a[f + g, j, k] // S;
a /: g_ * a[f_, j_, k_] := a[g f, j, k] // S;
c[0, _] := 0;
c /: c[f_, j_] + c[g_, j_] := c[f + g, j] // S;
c /: g_ * c[f_, j_] := c[g f, j] // S;
δa[0, _, _] := 0;
δa /: δa[f_, j_, k_] + δa[g_, j_, k_] := δa[f + g, j, k] // S;
δa /: g_ * δa[f_, j_, k_] := δa[g f, j, k] // S;
ca[0, _, _, _] := 0;
ca /: ca[f_, j_, k_, l_] + ca[g_, j_, k_, l_] := ca[f + g, j, k, l] // S;
ca /: g_ * ca[f_, j_, k_, l_] := ca[g f, j, k, l] // S;
δaa[0, _, _, _, _] := 0;
δaa /: δaa[f_, i_, j_, k_, l_] + δaa[g_, i_, j_, k_, l_] :=
  δaa[f + g, i, j, k, l] // S;
δaa /: g_ * δaa[f_, i_, j_, k_, l_] := δaa[g f, i, j, k, l] // S;
```

δaa relations

```
ϵ1 = 1;
```

Locality:

```
S[δaa[f_, i_, j_, k_, l_]] /; ({i, j} ⋂ {k, l} === {}) ⋀ ! OQ[i, k] :=
  δaa[f, k, l, i, j] // S;
```

Standard Swinging:

```
S[δaa[f_, i_, j_, k_, l_]] /; DQ[i, j, k, l] ⋀ OQ[i, k] ⋀ ! OQ[j, l] := S[Expand[
    δaa[f, i, l, k, j] +
     ϵ1 (ca[bk f, l, i, j] - ca[bi f, l, k, j] - ca[bk f, j, i, l] + ca[bi f, j, k, l])
   ]];
```

1322 Swinging:

```
S[δaa[f_, i_, j_, k_, k_]] /; DQ[i, j, k] ⋀ OQ[i, k, j] := S[Expand[
    δaa[f, i, k, k, j] +
     ϵ1 (ca[bk f, k, i, j] - ca[bi f, k, k, j] - ca[bk f, j, i, k] + ca[bi f, j, k, k]) +
     ϵ3 (δa[bi f, k, j] - c[bi bk, j])
   ]];
```

Tails Commute:

```
S[δaa[f_, i_, j_, i_, l_]] /; DQ[i, j, l] ⋀ ! OQ[j, l] := δaa[f, i, l, i, j] // S;
```

Commute Heads:

```
S[δaa[f_, i_, k_, j_, k_]] /; DQ[i, j, k] ⋀ ! OQ[i, j] := S[Expand[
    δaa[f, j, k, i, k] + ϵ2 (δa[bi f, j, k] - δa[bj f, i, k])
   ]];
```

Commute Heads/Tails:

```
S[δaa[f_, i_, j_, k_, i_]] /; DQ[i, j, k] ⋀ ! OQ[i, k] := S[
   δaa[f, k, i, i, j] + δaa[f, k, j, i, j] - δaa[f, i, j, k, j]
  ];
```

NonCommutativeMultiply

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[0, _] = 0; NonCommutativeMultiply[_, 0] = 0;
NonCommutativeMultiply[x_, x_] = 0;
NonCommutativeMultiply[x_Plus, y_] := NonCommutativeMultiply[#, y] & /@ x;
NonCommutativeMultiply[x_, y_Plus] := NonCommutativeMultiply[x, #] & /@ y;
```

```
β[f_] ** a[g_, j_, k_] := a[f g, j, k];
β[f_] ** c[g_, j_] := c[f g, j];
c[g_, j_] ** β[f_] := c[f g, j];
β[f_] ** δa[g_, j_, k_] := δa[f g, j, k];
β[f_] ** ca[g_, i_, j_, k_] := ca[f g, i, j, k];
β[f_] ** δaa[g_, i_, j_, k_, l_] := δaa[f g, i, j, k, l];
δa[g_, j_, k_] ** β[f_] := δa[f g, j, k];
δ ** a[f_, i_, j_] := δa[f, i, j];
c[f_, i_] ** a[g_, j_, k_] := ca[f g, i, j, k];
a[f_, i_, j_] ** δa[g_, k_, l_] := δaa[f g, i, j, k, l];
δa[f_, i_, j_] ** a[g_, k_, l_] := δaa[f g, i, j, k, l];
```

```
δ ** _c = 0;
δ ** _δa = 0;
δ ** _ca = 0;
δ ** _δaa = 0;
_c ** _c = 0;
_c ** _δa = _δa ** _c = 0;
_c ** _ca = _ca ** _c = 0;
_c ** _δaa = _δaa ** _c = 0;
_δa ** _δa = 0;
_δa ** _δaa = _δaa ** _δa = 0;
_δa ** _ca = _ca ** _δa = 0;
```

```
NonCommutativeMultiply::ndef =
  "NonCommuatativeMultiply is not defined on {`1`,`2`}."
NonCommutativeMultiply[x_, y_] :=
  (Message[NonCommutativeMultiply::ndef, x, y]; Undefined);
```

NonCommuatativeMultiply is not defined on {`1`,`2`}.

Bracket Generalities

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

The γ shortcuts

```
γ[f_, j_, k_] := δa[f, j, k] - c[b_j f, k] // S;
γ[f_, j_, k_, l_] /; DQ[j, k, l] := ca[f, l, j, k] - ca[f, k, j, l] // S;
```

Fundamental Brackets

a-β, a-c, a-a, AS

```
B[a[j_, k_], β[g_]] := γ[∂_{b_j} g - ∂_{b_k} g, j, k];
B[β[g_], a[j_, k_]] := -B[a[j, k], β[g]];
B[a[j_, k_], a[l_, m_]] /; ({j, k} ⋂ {l, m} === {}) := 0;
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // S;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // S;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := a[b_j, k, l] - a[b_k, j, l] - γ[1, j, k, l] // S;
B[a[k_, l_], a[j_, k_]] /; DQ[j, k, l] := -B[a[j, k], a[k, l]];
B[a[j_, k_], a[k_, j_]] /; DQ[j, k] :=
  a[b_j, k, j] - a[b_k, j, k] + a[b_j, k, k] - a[b_k, j, j] + ca[1, k, k, j] -
   ca[1, j, j, k] + ca[1, k, j, j] - ca[1, j, k, k] + γ[1, j, k] - γ[1, k, j];
B[a[j_, k_], a[j_, j_]] /; DQ[j, k] := -γ[1, j, k] // S;
B[a[j_, j_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
B[a[j_, k_], a[k_, k_]] /; DQ[j, k] := -γ[1, j, k] // S;
B[a[k_, k_], a[j_, k_]] /; DQ[j, k] := -B[a[j, k], a[j, j]];
B[a[f_, j_, k_], c[g_, j_]] /; DQ[j, k] := γ[-f g, j, k];
B[a[f_, j_, k_], c[g_, k_]] /; DQ[j, k] := γ[f g, j, k];
B[a[f_, j_, k_], c[g_, l_]] /; ({j, k} ⋂ {l} === {}) := 0;
B[a[f_, j_, j_], c[g_, j_]] = 0;
B[c[g_, l_], a[f_, j_, k_]] := -B[a[f, j, k], c[g, l]];
```

Vanishing brackets

```
B[_β, _β | δ | _c | _δa | _ca | _δaa] = 0;
B[_β | δ | _c | _δa | _ca | _δaa, _β] = 0;
B[δ | _c | _δa | _ca | _δaa, δ | _c | _δa | _ca | _δaa] = 0;
```

Composite Brackets

```
B[a[f_, j_, k_], β[g_]] := β[f] ** B[a[j, k], β[g]];
B[β[g_], a[f_, j_, k_]] := -B[a[f, j, k], β[g]];
B[a[f_, j_, k_], a[l_, m_]] :=
  B[β[f], a[l, m]] ** a[1, j, k] + β[f] ** B[a[j, k], a[l, m]];
B[a[f_, j_, k_], a[g_, l_, m_]] :=
  B[a[f, j, k], β[g]] ** a[1, l, m] + β[g] ** B[a[f, j, k], a[l, m]];
B[a[f_, i_, j_], δa[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[δa[f_, i_, j_], a[g_, k_, l_]] := δ ** B[a[f, i, j], a[g, k, l]];
B[a[f_, i_, j_], ca[g_, k_, l_, m_]] :=
  B[a[f, i, j], c[g, k]] ** a[1, l, m] + c[g, k] ** B[a[f, i, j], a[l, m]];
B[ca[g_, k_, l_, m_], a[f_, i_, j_]] := -B[a[f, i, j], ca[g, k, l, m]];
B[a[f_, i_, j_], δaa[g_, k_, l_, m_, n_]] :=
  B[a[f, i, j], δa[g, k, l]] ** a[1, m, n] + δa[g, k, l] ** B[a[f, i, j], a[m, n]];
B[δaa[g_, k_, l_, m_, n_], a[f_, i_, j_]] := -B[a[f, i, j], δaa[g, k, l, m, n]];
```

```
B::ndef = "B is not defined on {`1`,`2`}."
B[x_, y_] := (Message[B::ndef, x, y]; Undefined);
```

```
B is not defined on {`1`,`2`}.
```

Testing Jacobi and Anti-Symmetry

```
FormalBasis[n_, f_] := Module[{ff},
   ff = f@@Table[bi, {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, i, j], {i, n}, {j, n}],
      Table[c[ff, i], {i, n}],
      Table[δa[ff, i, j], {i, n}, {j, n}],
      Table[ca[ff, i, j, k], {i, n}, {j, n}, {k, n}],
      Table[δaa[ff, i, j, k, l], {i, n}, {j, n}, {k, i, n}, {l, j, n}]
     } /. 1[___] → 1
  ];
FormalPlusBasis[n_, f_] := Module[{ff},
   ff = f@@Table[bi, {i, n}];
   Flatten@{
      β[ff],
      Table[a[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[c[ff, i], {i, n}],
      Table[δa[ff, i, j], {i, n - 1}, {j, i + 1, n}],
      Table[ca[ff, i, j, k], {i, n}, {j, n - 1}, {k, j + 1, n}],
      Table[δaa[ff, i, j, k, l], {i, n - 1}, {j, i + 1, n}, {k, n - 1}, {l, k + 1, n}]
     } /. 1[___] → 1
  ];
```

```
FormalPlusBasis[3, f]
```

```
{β[f[b1, b2, b3]], a[f[b1, b2, b3], 1, 2], a[f[b1, b2, b3], 1, 3], a[f[b1, b2, b3], 2, 3],
 c[f[b1, b2, b3], 1], c[f[b1, b2, b3], 2], c[f[b1, b2, b3], 3], δa[f[b1, b2, b3], 1, 2],
 δa[f[b1, b2, b3], 1, 3], δa[f[b1, b2, b3], 2, 3], ca[f[b1, b2, b3], 1, 1, 2],
 ca[f[b1, b2, b3], 1, 1, 3], ca[f[b1, b2, b3], 1, 2, 3], ca[f[b1, b2, b3], 2, 1, 2],
 ca[f[b1, b2, b3], 2, 1, 3], ca[f[b1, b2, b3], 2, 2, 3], ca[f[b1, b2, b3], 3, 1, 2],
 ca[f[b1, b2, b3], 3, 1, 3], ca[f[b1, b2, b3], 3, 2, 3], δaa[f[b1, b2, b3], 1, 2, 1, 2],
 δaa[f[b1, b2, b3], 1, 2, 1, 3], δaa[f[b1, b2, b3], 1, 2, 2, 3],
 δaa[f[b1, b2, b3], 1, 3, 1, 2], δaa[f[b1, b2, b3], 1, 3, 1, 3],
 δaa[f[b1, b2, b3], 1, 3, 2, 3], δaa[f[b1, b2, b3], 2, 3, 1, 2],
 δaa[f[b1, b2, b3], 2, 3, 1, 3], δaa[f[b1, b2, b3], 2, 3, 2, 3]}
```

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalPlusBasis[3, f],
   FormalPlusBasis[3, g]
  ]], 0]
```

```
{}
```

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
DeleteCases[Flatten[Outer[
   AS,
   FormalBasis[3, f],
   FormalBasis[3, g]
  ]], 0]
```

```
{{a[f[b1, b2, b3], 1, 2], δaa[g[b1, b2, b3], 2, 2, 2, 3]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 1, 3], a[g[b1, b2, b3], 2, 2]} → c[-b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 1, 3], ca[g[b1, b2, b3], 1, 2, 2]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 1, 3], ca[g[b1, b2, b3], 3, 2, 2]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 1, 3], δaa[g[b1, b2, b3], 2, 1, 2, 2]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 1, 3], δaa[g[b1, b2, b3], 2, 2, 2, 3]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 2, 1], δaa[g[b1, b2, b3], 1, 2, 1, 3]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 2, 2], a[g[b1, b2, b3], 1, 3]} → c[-b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 2, 3], δaa[g[b1, b2, b3], 1, 2, 2, 2]} → c[-2 b1 b2 ∈3, 3],
 {a[f[b1, b2, b3], 3, 2], δaa[g[b1, b2, b3], 1, 3, 2, 3]} → c[-2 b1 b2 ∈3, 3],
 {ca[f[b1, b2, b3], 1, 2, 2], a[g[b1, b2, b3], 1, 3]} → c[-2 b1 b2 ∈3, 3],
 {ca[f[b1, b2, b3], 3, 2, 2], a[g[b1, b2, b3], 1, 3]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 1, 2, 1, 3], a[g[b1, b2, b3], 2, 1]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 1, 2, 2, 2], a[g[b1, b2, b3], 2, 3]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 1, 3, 2, 3], a[g[b1, b2, b3], 3, 2]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 2, 1, 2, 2], a[g[b1, b2, b3], 1, 3]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 2, 2, 2, 3], a[g[b1, b2, b3], 1, 2]} → c[-2 b1 b2 ∈3, 3],
 {δaa[f[b1, b2, b3], 2, 2, 2, 3], a[g[b1, b2, b3], 1, 3]} → c[-2 b1 b2 ∈3, 3]}
```

```
Jacobi[x1_, x2_, x3_] := Module[{Jac},
   Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
   If[Jac === 0, Jac, {x1, x2, x3} → Jac]
  ];
```

```
JacPlusErrors = DeleteCases[
  bas1 = FormalPlusBasis[4, f];
  bas2 = FormalPlusBasis[4, g];
  bas3 = FormalPlusBasis[4, h];
  Flatten[
   Table[Jacobi[bas1⟦i⟧, bas2⟦j⟧, bas3⟦k⟧],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

{}

```
JacPlusErrors // Length
```

0

```
JacErrors = DeleteCases[
  bas1 = FormalBasis[4, f];
  bas2 = FormalBasis[4, g];
  bas3 = FormalBasis[4, h];
  Flatten[
   Table[Jacobi[bas1[[i]], bas2[[j]], bas3[[k]]],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

B::ndef : B is not defined on {a[1, 1],a[1, 2]}.

NonCommutativeMultiply::ndef : NonCommuatativeMultiply is not defined on {β[g[b_1, b_2, b_3, b_4]],Undefined}. ≫

NonCommutativeMultiply::ndef : NonCommuatativeMultiply is not defined on {β[h[b_1, b_2, b_3, b_4]],Undefined}. ≫

B::ndef : B is not defined on {β[f[b_1, b_2, b_3, b_4]],Undefined}.

B::ndef : B is not defined on {a[g[b_1, b_2, b_3, b_4], 1, 1],c[h[b_1, b_2, b_3, b_4] b_1 ($f^{(0,1,0,0)}$[b_1, b_2, b_3, b_4] – $f^{(1,0,0,0)}$[b_1, b_2, b_3, b_4]), 2]}.

General::stop : Further output of B::ndef will be suppressed during this calculation. ≫

NonCommutativeMultiply::ndef : NonCommuatativeMultiply is not defined on {β[g[b_1, b_2, b_3, b_4]],Undefined}. ≫

General::stop : Further output of NonCommutativeMultiply::ndef will be suppressed during this calculation. ≫

Simplify::time :
Time spent on a transformation exceeded 300.` seconds, and the transformation was aborted. Increasing the value of TimeConstraint option may improve the result of simplification. ≫

```
{{β[f[b1, b2, b3, b4]], a[g[b1, b2, b3, b4], 1, 1], a[h[b1, b2, b3, b4], 1, 2]} →
  Undefined, {β[f[b1, b2, b3, b4]], a[g[b1, b2, b3, b4], 1, 1],
   a[h[b1, b2, b3, b4], 1, 3]} → Undefined, ⋯ 456 219 ⋯,
 {a[f[b1, b2, b3, b4], 4, 4], δaa[g[b1, b2, b3, b4], 4, 4, 4, 4],
   δaa[h[b1, b2, b3, b4], 4, 3, 4, 4]} → Undefined}
```

large output | show less | show more | show all | set size limit...

```
JacErrors // Length
```

456 222

The Adjoint action

AutoAd

```
AutoAd[x_][y_] :=
  Module[{pows, states, i, s, seq, sh = 5, dseq, sf1, sf2, sf, t1, n},
   pows = NestList[B[x, #] &, y, 20];
   Print["pows computed for ", {x, y}, "..."];
   states = Union[
     Cases[pows, s_β | s_a | s_c | s_δa | s_ca | s_δaa ⧴ ReplacePart[s, 1 → _], ∞]];
   Sum[
    seq = Cases[{#}, states〚i〛, ∞] & /@ pows;
    seq = Replace[seq, {{_[f_, ___]} ⧴ f, {} → 0}, {1}];
    Print["seq computed... ", states〚i〛, " is ", i, "/", Length@states];
    dseq = Drop[seq, sh];
    If[Union[Length[MonomialList[#]] & /@ dseq] === {1} ⋀
      Union[Length[FactorTermsList[#]] & /@ dseq] === {2},
     sf1 = FindSequenceFunction[FactorTermsList[#]〚1〛 & /@ dseq];
     sf2 = FindSequenceFunction[FactorTermsList[#]〚2〛 & /@ dseq];
     Print["sf1: ", sf1, " sf2: ", sf2];
     sf = (sf1[#] sf2[#] &),
     (*Else*) sf = FindSequenceFunction[dseq,
       FunctionSpace → {"ConstantRecursive", "HolonomicSequence",
         "Polynomial", "RationalFunction", "HypergeometricTerm"}];
     Print["sf: ", sf];
    ];
    ReplacePart[states〚i〛, 1 → Simplify[Sum_{n=0}^{sh-1} seq〚n + 1〛/n! + Sum_{n=sh}^{∞} sf[n + 1 - sh]/n!]],
    {i, Length@states}]
  ];
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)

AutoAd[a[t, j, k]][β[f[b_j, b_k]]]
```

```
pows computed for {a[t, j, k], β[f[b_j, b_k]]}...
seq computed... c[_, k] is 1/3
sf: -t^4 b_j^4 (-t b_j)^#1 (f^(0,1)[b_j, b_k] - f^(1,0)[b_j, b_k]) &
seq computed... β[_] is 2/3
sf1: 1 & sf2: 0 &
seq computed... δa[_, j, k] is 3/3
sf: t^4 b_j^3 (-t b_j)^#1 (f^(0,1)[b_j, b_k] - f^(1,0)[b_j, b_k]) &
```

$$c\left[e^{-t b_j}\left(-1+e^{t b_j}\right)\left(f^{(0,1)}[b_j, b_k]-f^{(1,0)}[b_j, b_k]\right), k\right] + \beta[f[b_j, b_k]] + \delta a\left[-\frac{e^{-t b_j}\left(-1+e^{t b_j}\right)\left(f^{(0,1)}[b_j, b_k]-f^{(1,0)}[b_j, b_k]\right)}{b_j}, j, k\right]$$

Ad

Ad[a[t_, j_, k_]][β[f_]] /; FreeQ[t, b_] :=

$$\beta[f] + c\left[\left(1-e^{-t b_j}\right)(\partial_{b_k} f - \partial_{b_j} f), k\right] + \delta a\left[\frac{\left(e^{-t b_j}-1\right)(\partial_{b_k} f - \partial_{b_j} f)}{b_j}, j, k\right];$$

Ad[a[t_, j_, k_]][a[1, j_, k_]] /; FreeQ[t, b_] := a[1, j, k];

Ad[a[t_, j_, k_]][a[1, n_, i_]] /; FreeQ[t, b_] ⋀ ({j, k} ⋂ {n, i} === {}) := a[1, n, i];

Ad[a[t_, j_, k_]][a[1, i_, j_]] /; DQ[i, j, k] ⋀ FreeQ[t, b_] :=

$$a[1, i, j] + a\left[1-e^{-t b_j}, i, k\right] + a\left[\frac{\left(e^{-t b_j}-1\right) b_i}{b_j}, j, k\right] + ca\left[\frac{1-e^{-t b_j}}{b_j}, k, i, j\right] +$$
$$ca\left[\frac{e^{-t b_j}-1}{b_j}, j, i, k\right] + ca\left[\frac{b_i\left(1-e^{-t b_j}-t b_j\right)}{b_j^2}, j, j, k\right] +$$
$$ca\left[\frac{e^{-2 t b_j} b_i\left(1-e^{t b_j}-e^{t b_j}\left(e^{t b_j}-2\right) t b_j\right)}{b_j^2}, k, j, k\right] + ca\left[\frac{e^{-2 t b_j}\left(e^{t b_j}\left(1-t b_j\right)-1\right)}{b_j},\right.$$
$$\left. k, i, k\right] + \delta a\left[\frac{b_i\left(-1+e^{-t b_j}+t b_j\right)\epsilon_2}{b_j^2} + \frac{b_i\left(1-e^{-2 t b_j}+\left(-1-e^{-t b_j}\right) t b_j\right)\epsilon_2}{b_j^2}, j, k\right] +$$
$$\delta a\left[-\frac{\left(-1+e^{-t b_j}+t b_j\right)\epsilon_2}{b_j} - \frac{\left(1-e^{-2 t b_j}+\left(-1-e^{-t b_j}\right) t b_j\right)\epsilon_2}{b_j}, i, k\right] +$$
$$\delta aa\left[\frac{2 e^{-t b_j} b_i (\mathrm{Sinh}[t b_j]-t b_j)}{b_j^3}, j, k, j, k\right] + \delta aa\left[\frac{-1+e^{-t b_j}+t b_j}{b_j^2}, i, j, j, k\right] +$$
$$\delta aa\left[-\frac{1-e^{-2 t b_j}+\left(-1-e^{-t b_j}\right) t b_j}{b_j^2}, i, k, j, k\right];$$

Ad[a[t_, j_, k_]][a[1, i_, k_]] /; DQ[i, j, k] ⋀ FreeQ[t, b_] :=

$$a\left[e^{-t b_j}, i, k\right] + a\left[\frac{\left(1-e^{-t b_j}\right) b_i}{b_j}, j, k\right] + ca\left[\frac{2 e^{-t b_j} b_i (\mathrm{Sinh}[t b_j]-t b_j)}{b_j^2}, k, j, k\right] +$$
$$ca\left[\frac{e^{-2 t b_j}\left(1+e^{t b_j}\left(-1+t b_j\right)\right)}{b_j}, k, i, k\right] + \delta a\left[-\frac{e^{-2 t b_j} b_i\left(-1+e^{t b_j}\left(1-t b_j\right)\right)\epsilon_2}{b_j^2}, j, k\right] +$$

```
δa[(e^(-2 t b_j) (-1 + e^(t b_j) (1 - t b_j)) ϵ2)/b_j, i, k] + δaa[(2 e^(-t b_j) b_i (-Sinh[t b_j] + t b_j))/b_j^3,
   j, k, j, k] + δaa[(e^(-2 t b_j) (-1 + e^(t b_j) (1 - t b_j)))/b_j^2, i, k, j, k];
Ad[a[t_, j_, k_]][a[1, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[1, j, l] + ca[t, l, j, k] + ca[(e^(-t b_j) - 1)/b_j, k, j, l] + δaa[(1 - e^(-t b_j) - t b_j)/b_j^2, j, k, j, l];
Ad[a[t_, j_, k_]][a[1, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[e^(t b_j), k, l] + a[-((-1 + e^(t b_j)) b_k)/b_j, j, l] + ca[(-1 + e^(t b_j) (1 - t b_j))/b_j, k, k, l] +
   ca[(b_j - e^(-t b_j) b_j + b_k + e^(t b_j) (-1 + t b_j) b_k)/b_j^2, k, j, l] +
   ca[(b_j + b_k + t b_j b_k - e^(t b_j) (b_j + b_k))/b_j^2, l, j, k] + δaa[(1 + e^(t b_j) (-1 + t b_j))/b_j^2, j, k, k, l] +
   δaa[1/b_j^3 e^(-t b_j) (b_j + e^(2 t b_j) (b_j + (2 - t b_j) b_k) - e^(t b_j) (2 b_k + b_j (2 + t b_k))), j, k, j, l];
Ad[a[t_, j_, k_]][c[1, i_]] /; FreeQ[t, b_] ∧ ({j, k} ⋂ {i} === {}) := c[1, i];
Ad[a[t_, j_, k_]][c[1, j_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[1, j] + c[1 - e^(-t b_j), k] + δa[(e^(-t b_j) - 1)/b_j, j, k];
Ad[a[t_, j_, k_]][c[1, k_]] /; DQ[j, k] ∧ FreeQ[t, b_] :=
  c[e^(-t b_j), k] + δa[(1 - e^(-t b_j))/b_j, j, k];

Ad[x_β | x_c | x_δa | x_ca | x_δaa][y_] := y + B[x, y];

Ad[x_][a[f_, i_, j_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][a[1, i, j]];
Ad[x_][c[f_, i_]] /; f =!= 1 := Ad[x][β[f]] ** Ad[x][c[1, i]];
Ad[x_][δa[f_, j_, k_]] := δ ** (β[f] ** Ad[x][a[1, j, k]]);
Ad[x_][ca[f_, i_, j_, k_]] := Ad[x][c[f, i]] ** Ad[x][a[1, j, k]];
Ad[x_][δaa[f_, i_, j_, k_, l_]] := Ad[x][δa[f, i, j]] ** Ad[x][a[1, k, l]];
Ad[x_][y_Plus] := Ad[x] /@ y;

Ad::ndef = "Ad[`1` is not defined on `2`.";
Ad[x_][y_] := (Message[Ad::ndef, x, y]; Undefined);
```

AutoAd - Ad tests

```
Module[{t1, t2},
   {t1 = S[AutoAd[a[t, j, k]][#]],
    S[Ad[a[t, j, k]][#] - t1]}
  ] & @ a[1, i, k]
```

$$\left\{a\left[e^{-t b_j}, i, k\right] + a\left[\frac{\left(1-e^{-t b_j}\right) b_i}{b_j}, j, k\right] + \mathrm{ca}\left[\frac{e^{-2 t b_j} b_i \left(-1+e^{2 t b_j}-2 e^{t b_j} t b_j\right)}{b_j^2}, k, j, k\right] + \mathrm{ca}\left[\frac{e^{-2 t b_j} \left(1-e^{t b_j}+e^{t b_j} t b_j\right)}{b_j}, k, i, k\right] + \delta a\left[\frac{e^{-2 t b_j} b_i \left(1-e^{t b_j}+e^{t b_j} t b_j\right) \epsilon_2}{b_j^2}, j, k\right] + \delta a\left[-\frac{e^{-2 t b_j} \left(1-e^{t b_j}+e^{t b_j} t b_j\right) \epsilon_2}{b_j}, i, k\right] + \delta aa\left[-\frac{e^{-2 t b_j} b_i \left(-1+e^{2 t b_j}-2 e^{t b_j} t b_j\right)}{b_j^3}, j, k, j, k\right] + \delta aa\left[\frac{e^{-2 t b_j} \left(-1+e^{t b_j}-e^{t b_j} t b_j\right)}{b_j^2}, i, k, j, k\right], 0\right\}$$

```
AdTests[a[t, j, k]] = {β[f[bj, bk]], a[1, j, k], a[1, n, i], c[1, i],
   c[1, j], c[1, k], a[1, j, l], a[1, i, j], a[1, i, k], a[1, k, l]};
```

```
S[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ Take[AdTests[a[t, j, k]], All]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

The semi group properties

```
Module[{t1, t2, t3, t4},
    t1 = Ad[a[t, j, k]][#] /.
      (h : (β | a | c | δa | ca | δaa))[c_, r___] :> h[SeriesCoefficient[c, {t, 0, 1}], r];
    t2 = B[a[1, j, k], #];
    t3 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
    t4 = # // Ad[a[t + s, j, k]];
    # → S[{t1 == t2, t3 - t4}]
   ] & /@ AdTests[a[t, j, k]] // ColumnForm
```

$\beta[f[b_j, b_k]] \to \{\mathrm{True}, 0\}$
a[1, j, k] → {True, 0}
a[1, n, i] → {True, 0}
c[1, i] → {True, 0}
c[1, j] → {True, 0}
c[1, k] → {True, 0}
a[1, j, l] → {True, 0}
a[1, i, j] → {True, 0}
a[1, i, k] → {True, 0}
a[1, k, l] → {True, 0}

R

```
Switch[6,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t bᵢ, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[bᵢ, bⱼ]]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bⱼ] + bᵢ f1[bⱼ]]],
  4, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[c[f3[bᵢ, bⱼ], i]] // Ad[c[f4[bᵢ, bⱼ], j]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f5[bᵢ], i, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[ca[f6[bᵢ, bⱼ], j, i, j]],
  7, R[i_, j_][x_] :=
   x // Ad[a[1, i, j]] // Ad[ca[f5[bᵢ], i, i, j]] // Ad[ca[f6[bᵢ, bⱼ], j, i, j]]
 ];
VerifyR3[expr_] := Module[{lhs, rhs}, {
   lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // S;
   rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // S;
   S[lhs - rhs] == 0
  }]
```

Verifying R3

```
VerifyR3 /@ {β[f[b₁, b₂, b₃, b₄]], c[f[b₁, b₂, b₃, b₄], 1], c[f[b₁, b₂, b₃, b₄], 2],
  c[f[b₁, b₂, b₃, b₄], 3], δa[f[b₁, b₂, b₃, b₄], 1, 2], δa[f[b₁, b₂, b₃, b₄], 1, 3],
  δa[f[b₁, b₂, b₃, b₄], 2, 3], a[f[b₁, b₂, b₃, b₄], 1, 4], a[f, 2, 4]}
```

{{True}, {True}, {True}, {True}, {True}, {True}, {True}, {True},

$$\Big\{\mathrm{ca}\Big[2\, e^{b_1-b_2}\left(-1+e^{b_2}\right) f\, \mathrm{f6}[b_1, b_2]\, b_1, 4, 2, 3\Big] + \mathrm{ca}\Big[-\frac{e^{-b_2}\left(-1+e^{b_2}\right) f\left(2-2\,e^{b_1}+\left(1+e^{b_1}\right) b_1\right)}{b_1}, 3, 2, 4\Big] + \mathrm{ca}\Big[-2\, e^{b_1-b_2}\left(-1+e^{b_2}\right) f\, \mathrm{f6}[b_1, b_2]\, b_2, 4, 1, 3\Big] +$$

$$\mathrm{ca}\Big[\frac{e^{-b_2}\left(-1+e^{b_2}\right) f\left(2-2\,e^{b_1}+\left(1+e^{b_1}\right) b_1\right) b_2}{b_1^2}, 3, 1, 4\Big] +$$

$$\delta\mathrm{aa}\Big[\frac{1}{b_1^2} e^{-b_2}\left(-1+e^{b_2}\right) f\left(2-2\,e^{b_1}+\left(1+e^{b_1}\right) b_1 + 2\,e^{b_1}\,\mathrm{f6}[b_1, b_2]\, b_1^2\right), 1, 3, 2, 4\Big] +$$

$$\delta\mathrm{aa}\Big[-\frac{2\, e^{b_1-b_2}\left(-1+e^{b_2}\right) f\, \mathrm{f6}[b_1, b_2]\, b_1}{b_2}, 2, 3, 2, 4\Big] +$$

$$\delta\mathrm{aa}\Big[-\frac{e^{-b_2}\left(-1+e^{b_2}\right) f\left(2-2\,e^{b_1}+\left(1+e^{b_1}\right) b_1\right) b_2}{b_1^3}, 1, 3, 1, 4\Big] == 0\Big\}\Big\}$$