Pensieve header: Some Fox-Milnor Examples.

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SetDirectory["C:/drorbn/AcademicPensieve/2015-07/"]
```

```
C:\drorbn\AcademicPensieve\2015-07\PolyPoly
```

Initialization

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ ⧴ a, ∞];
   M = Outer[Factor[∂h#1 t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
```

Program

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
```

$$\mathbf{m}_{a\_b\_\to c\_}[\Gamma[\omega\_,\lambda\_]] := \mathrm{Module}\Big[\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu\},$$

$$\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix} = \begin{pmatrix}\partial_{t_a,h_a}\lambda & \partial_{t_a,h_b}\lambda & \partial_{t_a}\lambda\\ \partial_{t_b,h_a}\lambda & \partial_{t_b,h_b}\lambda & \partial_{t_b}\lambda\\ \partial_{h_a}\lambda & \partial_{h_b}\lambda & \lambda\end{pmatrix} \;/.\; (t\,|\,h)_{a|b}\to 0;$$

$$\Gamma\Big[(\mu = 1-\beta)\,\omega,\ \{t_c, 1\}.\begin{pmatrix}\gamma+\alpha\,\delta/\mu & \epsilon+\delta\,\theta/\mu\\ \phi+\alpha\,\psi/\mu & \Xi+\psi\,\theta/\mu\end{pmatrix}.\{h_c, 1\}\Big]$$

$$/.\ \{T_a\to T_c,\ T_b\to T_c\}\ //\ \Gamma\mathrm{Collect}\Big];$$

$$\mathbf{Rp}_{a\_b\_} := \Gamma\Big[1,\ \{t_a, t_b\}.\begin{pmatrix}1 & 1-T_a\\ 0 & T_a\end{pmatrix}.\{h_a, h_b\}\Big];$$

$$\mathbf{Rm}_{a\_b\_} := \mathbf{Rp}_{ab}\ /.\ T_a\to 1\big/T_a;$$

MetaAssoc

$$\eta = \Gamma\Big[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_S\}\Big];$$

$$(\eta\ //\ \mathbf{m}_{12\to1}\ //\ \mathbf{m}_{13\to1}) == (\eta\ //\ \mathbf{m}_{23\to2}\ //\ \mathbf{m}_{12\to1})$$

MetaAssoc

```
True
```

R3

$$\{\mathbf{Rm}_{51}\,\mathbf{Rm}_{62}\,\mathbf{Rp}_{34}\ //\ \mathbf{m}_{14\to1}\ //\ \mathbf{m}_{25\to2}\ //\ \mathbf{m}_{36\to3},$$
$$\mathbf{Rp}_{61}\,\mathbf{Rm}_{24}\,\mathbf{Rm}_{35}\ //\ \mathbf{m}_{14\to1}\ //\ \mathbf{m}_{25\to2}\ //\ \mathbf{m}_{36\to3}\}$$

R3

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

8_17

$$\xi = \mathbf{Rm}_{12,1}\,\mathbf{Rm}_{27}\,\mathbf{Rm}_{83}\,\mathbf{Rm}_{4,11}\,\mathbf{Rp}_{16,5}\,\mathbf{Rp}_{6,13}\,\mathbf{Rp}_{14,9}\,\mathbf{Rp}_{10,15};$$
$$\mathrm{Do}[\xi = \xi\ //\ \mathbf{m}_{1k\to1},\ \{k, 2, 16\}];\ \xi$$

8_17

$$\begin{pmatrix}11-\frac{1}{T_1^3}+\frac{4}{T_1^2}-\frac{8}{T_1}-8\,T_1+4\,T_1^2-T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$

tr

$$\mathbf{tr}_{a\_}[\Gamma[\omega\_,\lambda\_]] := \mathbf{Module}\Big[\{\alpha,\theta,\psi,\Xi\},\;\begin{pmatrix}\alpha & \theta\\ \psi & \Xi\end{pmatrix} = \begin{pmatrix}\partial_{t_a,h_a}\lambda & \partial_{t_a}\lambda\\ \partial_{h_a}\lambda & \lambda\end{pmatrix} \;/.\; (t\,|\,h)_a \to 0;\;\; \Gamma\big[\omega\,(1-\alpha),\ \Xi+\psi\star\theta\big/(1-\alpha)\big] \;//\; \Gamma\mathbf{Collect}\Big];$$

$$(\eta\;//\;m_{12\to1}\;//\;\mathbf{tr}_1) == (\eta\;//\;m_{21\to1}\;//\;\mathbf{tr}_1)$$

tr

True

$$(\eta\;//\;m_{12\to1}\;//\;\mathbf{tr}_1)$$

$$\begin{pmatrix} \omega\,(1+\alpha_{12}\,(-1+\alpha_{21})-\alpha_{21}-\alpha_{11}\,\alpha_{22}) & h_3\\ t_3 & \frac{\alpha_{13}\,\alpha_{22}\,\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{13}\,\alpha_{32}-\alpha_{13}\,\alpha_{21}\,\alpha_{32}+\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{33}-\alpha_{12}\,\alpha_{33}-\alpha_{21}\,\alpha_{33}+\alpha_{1}}{1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}\\ t_S & \frac{\alpha_{13}\,\alpha_{22}\,\phi_1+\alpha_{23}\,\phi_1-\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{13}\,\phi_2-\alpha_{13}\,\alpha_{21}\,\phi_2+\alpha_{11}\,\alpha_{23}\,\phi_2+\phi_3-\alpha_{12}\,\phi_3-\alpha_{21}\,\phi_3+\alpha_{12}\,\alpha}{1-\alpha_{12}-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}} \end{pmatrix}$$

$$\Big\{\varsigma = \Gamma\Big[\omega,\ \{t_1,t_2\}.\begin{pmatrix}1-\gamma & \beta\\ \gamma & 1-\beta\end{pmatrix}.\{h_1,h_2\}\Big],\ \varsigma\;//\;\mathbf{tr}_1,\ \varsigma\;//\;m_{12\to1}\Big\}$$

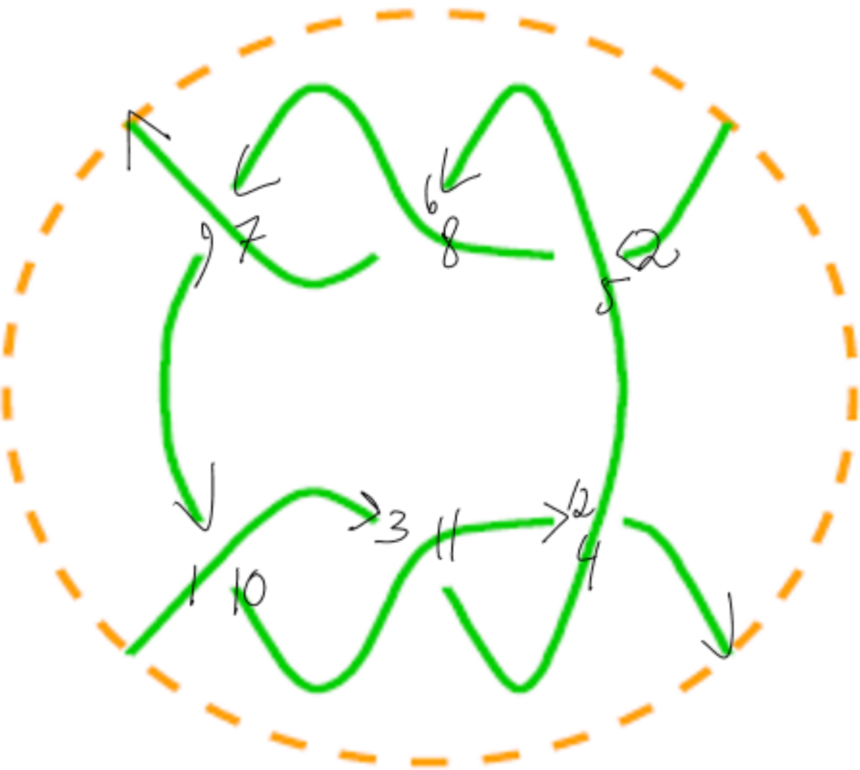
$$\left\{\begin{pmatrix}\omega & h_1 & h_2\\ t_1 & 1-\gamma & \beta\\ t_2 & \gamma & 1-\beta\end{pmatrix},\ \begin{pmatrix}\gamma\,\omega & h_2\\ t_2 & 1\end{pmatrix},\ \begin{pmatrix}\omega-\beta\,\omega & h_1\\ t_1 & 1\end{pmatrix}\right\}$$

$$\{\varsigma\,Rm_{34}\,Rp_{56}\;//\;m_{46\to0}\;//\;m_{31\to1}\;//\;m_{25\to2},\ \varsigma\,Rm_{34}\,Rp_{56}\;//\;m_{46\to0}\;//\;m_{13\to1}\;//\;m_{52\to2}\}$$

$$\left\{\begin{pmatrix}\omega & h_0 & h_1 & h_2\\ t_0 & \frac{T_2}{T_1} & 0 & 0\\ t_1 & -\frac{(-1+\gamma)\,(-1+T_1)}{T_1} & 1-\gamma & \beta\\ t_2 & \frac{1-\gamma+\gamma\,T_1-T_2}{T_1} & \gamma & 1-\beta\end{pmatrix},\ \begin{pmatrix}\omega & h_0 & h_1 & h_2\\ t_0 & \frac{T_2}{T_1} & 0 & 0\\ t_1 & \frac{-1+\beta+T_1-\beta\,T_2}{T_1} & 1-\gamma & \beta\\ t_2 & \frac{(-1+\beta)\,(-1+T_2)}{T_1} & \gamma & 1-\beta\end{pmatrix}\right\}$$

$$\left\{-\frac{(-1+\gamma)\,(-1+T_1)}{T_1} == \frac{-1+\beta+T_1-\beta\,T_2}{T_1},\ \frac{1-\gamma+\gamma\,T_1-T_2}{T_1} == \frac{(-1+\beta)\,(-1+T_2)}{T_1}\right\}\;//\;\mathbf{Simplify}$$

$$\left\{\frac{\beta-\gamma+\gamma\,T_1-\beta\,T_2}{T_1} == 0,\ \frac{\beta-\gamma+\gamma\,T_1-\beta\,T_2}{T_1} == 0\right\}$$



$\{\xi 1 = \mathrm{Rm}_{1,10}\,\mathrm{Rm}_{11,3}\,\mathrm{Rm}_{4,12}\,\mathrm{Rp}_{5,2}\,\mathrm{Rp}_{8,6}\,\mathrm{Rp}_{7,9}$ // $m_{1,3\to 1}$ // $m_{1,4\to 1}$ // $m_{1,5\to 1}$ // $m_{1,6\to 1}$ // $m_{1,7\to 1}$ // $m_{2,8\to 2}$ // $m_{2,9\to 2}$ // $m_{2,10\to 2}$ // $m_{2,11\to 2}$ // $m_{2,12\to 2}$, $\xi 1$ // $\mathrm{tr}_1\}$

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

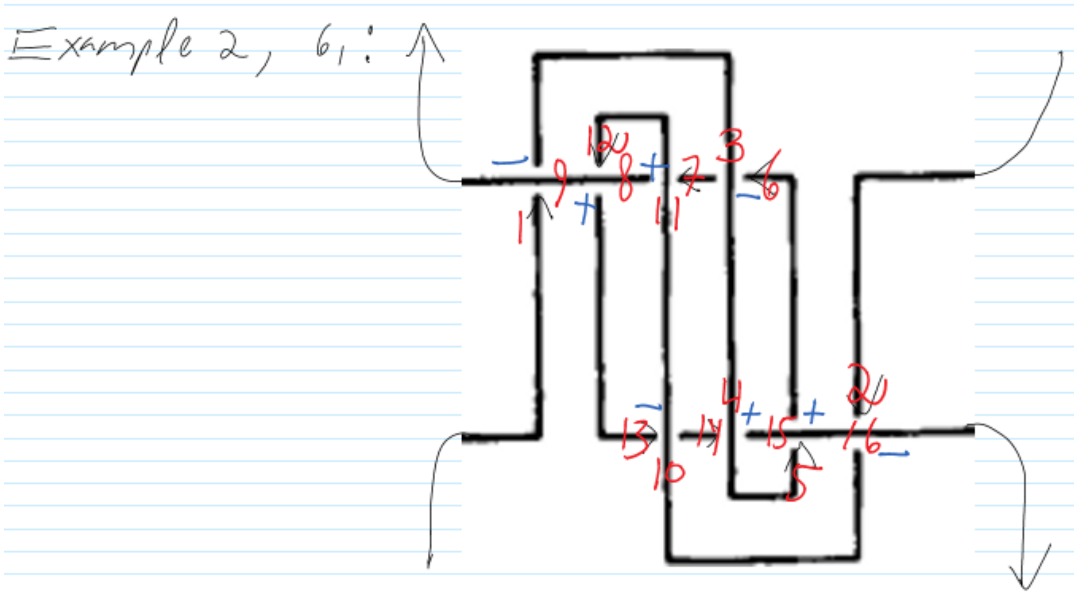
$$\left\{\begin{pmatrix} \frac{(1+T_1(-1+T_2))(1+(-1+T_1)T_2)}{T_1 T_2} & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}\right\}$$

$(1 + T_1 (-1 + T_2)) * ((1 + T_1 (-1 + T_2))$ /. $T_{i\_} :\to 1/T_i)$ // Factor

$$\frac{(1 - T_1 + T_1 T_2)(1 - T_2 + T_1 T_2)}{T_1 T_2}$$

$(1 + T_1 (-1 + T_2)) (1 + (-1 + T_1) T_2)$ // Factor

$(1 - T_1 + T_1 T_2)(1 - T_2 + T_1 T_2)$



ζ1 =

$$\mathrm{Rm}_{9,1}\,\mathrm{Rp}_{8,12}\,\mathrm{Rp}_{11,7}\,\mathrm{Rm}_{3,6}\,\mathrm{Rm}_{10,13}\,\mathrm{Rp}_{4,14}\,\mathrm{Rp}_{15,5}\,\mathrm{Rm}_{16,2}\ //\ m_{1,3\to1}\ //\ m_{1,4\to1}\ //\ m_{1,5\to1}\ //\ m_{1,6\to1}\ //\ m_{1,7\to1}\ //\ m_{1,8\to1}\ //\ m_{1,9\to1}\ //\ m_{2,10\to2}\ //\ m_{2,11\to2}\ //\ m_{2,12\to2}\ //\ m_{2,13\to2}\ //\ m_{2,14\to2}\ //\ m_{2,15\to2}\ //\ m_{2,16\to2}$$

$$\begin{pmatrix} 1+T_1\left(-1+\frac{1}{T_2}\right)-T_2+\frac{T_2}{T_1} & h_1 & h_2 \\ t_1 & \frac{T_1-T_1^2+T_2-2T_1T_2+T_1^2T_2-2T_2^2+T_1T_2^2}{-T_1^2-T_1T_2+T_1^2T_2-T_2^2+T_1T_2^2} & \frac{(-1+T_1)(T_1+T_2)}{-T_1^2-T_1T_2+T_1^2T_2-T_2^2+T_1T_2^2} \\ t_2 & \frac{(-1+T_2)(T_1+T_2)}{-T_1^2-T_1T_2+T_1^2T_2-T_2^2+T_1T_2^2} & \frac{T_1-2T_1^2+T_2-2T_1T_2+T_1^2T_2-T_2^2+T_1T_2^2}{-T_1^2-T_1T_2+T_1^2T_2-T_2^2+T_1T_2^2} \end{pmatrix}$$

ζ1 // $m_{1,2\to1}$

$$\begin{pmatrix} 5-\frac{2}{T_1}-2T_1 & h_1 \\ t_1 & 1 \end{pmatrix}$$

Factor$\left[5-\frac{2}{T_1}-2T_1\right]$

$$-\frac{(-2+T_1)(-1+2T_1)}{T_1}$$

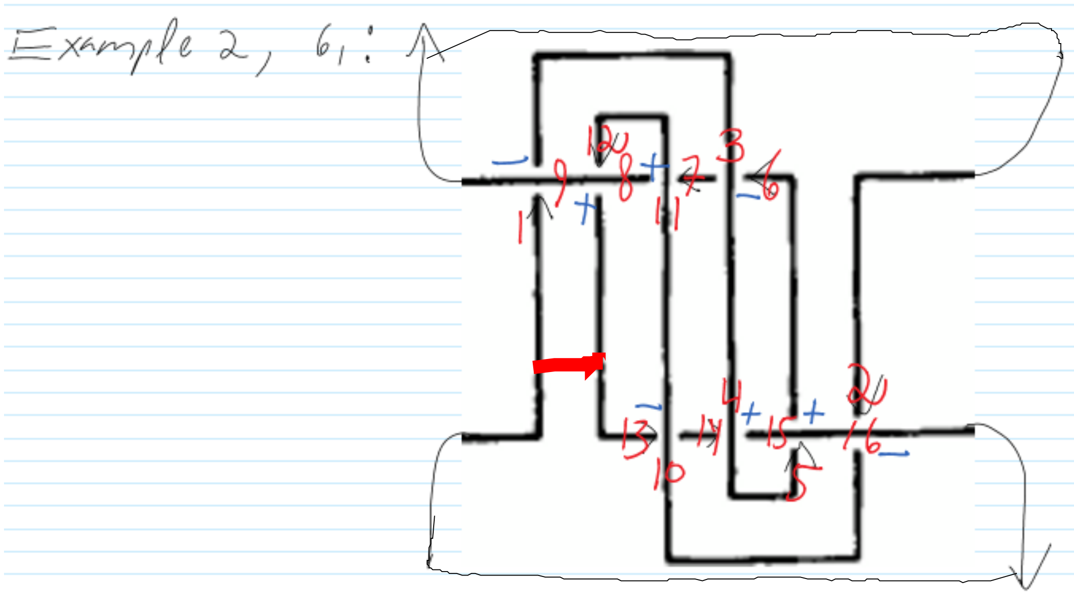
ζ1 // tr_1

$$\begin{pmatrix} -1+\frac{1-T_2}{T_1}+\frac{1}{T_2} & h_2 \\ t_2 & 1 \end{pmatrix}$$

$-1+\frac{1-T_2}{T_1}+\frac{1}{T_2}$ // Factor

$$-\frac{(-1+T_2)(T_1+T_2)}{T_1T_2}$$

6_1 done right:




```
ζ2 =
  Rm9,1 Rp8,12 Rp11,7 Rm3,6 Rm10,13 Rp4,14 Rp15,5 Rm16,2 // m1,3→1 // m1,4→1 // m1,5→1 // m1,6→1 //
        m1,7→1 // m1,8→1 // m1,9→1 // m2,10→2 //
      m2,11→2 // m2,12→2 // m1,2→1 // m13,14→2 // m2,15→2 // m2,16→2
```

$$\begin{pmatrix} \left(1+\frac{-1+T_1}{T_2}\right)\left(1+\left(-1+\frac{1}{T_1}\right)T_2\right) & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}$$

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ζ2 // m1,2→1
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$$\begin{pmatrix} 5-\frac{2}{T_1}-2\,T_1 & h_1 \\ t_1 & 1 \end{pmatrix}$$

$$\mathbf{Factor}\left[\left(1+\frac{-1+T_1}{T_2}\right)\left(1+\left(-1+\frac{1}{T_1}\right)T_2\right)\right]$$

$$-\frac{(-1+T_1+T_2)\,(-T_1-T_2+T_1 T_2)}{T_1 T_2}$$

$$\mathbf{Factor}\left[\left(1+\frac{-1+T_1}{T_2}\right)\left(1+\left(-1+\frac{1}{T_1}\right)T_2\right)\ /.\ T_2\to T_1\right]$$

$$-\frac{(-2+T_1)\,(-1+2\,T_1)}{T_1}$$