Cheat Sheet OneCo

Road Map. • Fully analyze and implement $\mathcal{L}^{1co}/2D$; verify Jacobi. • Compute Ad and solve for R. • Implement and verify scatter-level stitching. • Guess/deduce glow level.

Deriving Gassner. $\mathcal{L}^{2Dw}$ is $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo locality, $[a_{ij}, a_{ik}] = 0$, $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$, and (mod $\langle a_{ii}\rangle$) $[a_{ij}, a_{ji}] = b_i a_{ji} - b_j a_{ij}$. Acts on $V = \mathbb{Q}[\![b_i]\!]\langle x_i = a_{i\infty}\rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_i x_j - b_j x_i$. Hence $e^{\operatorname{ad} a_{ij}} x_i = x_i$, $e^{\operatorname{ad} a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i}(1 - e^{b_i}) x_i$. Renaming $\bar{x}_i = x_i/b_i$, $t_i = e^{b_i}$, get $[e^{\operatorname{ad} a_{ij}}]_{\bar{x}_i, \bar{x}_j} = \begin{pmatrix} 1 & 1 - t_i \\ 0 & t_i \end{pmatrix}$.

The $\mathcal{L}^{2Dw}$ Adjoint representation. $e^{\operatorname{ad} a_{ij}}$ acts by

$$a_{kl} \mapsto a_{kl}, \qquad a_{ik} \mapsto a_{ik}, \qquad a_{kj} \mapsto e^{-b_i} a_{kj} + \frac{b_k}{b_i}(1 - e^{-b_i}) a_{ij},$$

$$a_{ki} \mapsto a_{ki} + (1 - e^{-b_i}) a_{kj} + b_k \frac{e^{-b_i} - 1}{b_i} a_{ij},$$

$$a_{jk} \mapsto e^{b_i} a_{jk} + \frac{b_j}{b_i}(1 - e^{b_i}) a_{ik}, \quad a_{ji} \mapsto e^{b_i} a_{ji} + \frac{b_j}{b_i}(1 - e^{b_i}) a_{ij}.$$

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf.

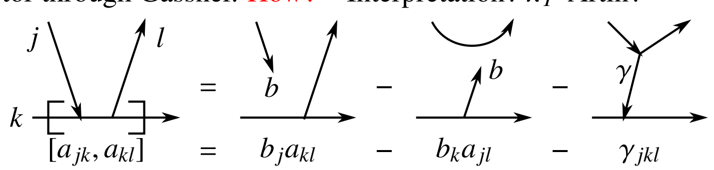
Adjoint Gassner. Renaming $\bar{a}_{ij} = a_{ij}/b_i$ and $t_i = e^{b_i}$, get $[\bar{a}_{ij}, \bar{a}_{ik}] = 0$, $[\bar{a}_{ik}, \bar{a}_{jk}] = -[\bar{a}_{ij}, \bar{a}_{jk}] = \bar{a}_{ik} - \bar{a}_{jk}$, and (mod $\langle \bar{a}_{ii}\rangle$) $[\bar{a}_{ij}, \bar{a}_{ji}] = \bar{a}_{ji} - \bar{a}_{ij}$, so

$$\bar{a}_{kj} \mapsto t_i^{-1} \bar{a}_{kj} + (1 - t_i^{-1}) \bar{a}_{ij},$$

$$\bar{a}_{ki} \mapsto \bar{a}_{ki} + (1 - t_i^{-1}) \bar{a}_{kj} + (t_i^{-1} - 1) \bar{a}_{ij},$$

$$\bar{a}_{jk} \mapsto t_i \bar{a}_{jk} + (1 - t_i) \bar{a}_{ik}, \quad \bar{a}_{ji} \mapsto t_i \bar{a}_{ji} + (1 - t_i) \bar{a}_{ij}.$$

Questions. • As Gassner is Γ calculus, Adjoint Gassner must factor through Gassner. How? • Interpretation? π_T-Artin?



2Dv. b: bracket trace; c: cobracket trace; $\langle b, c\rangle = \delta \in \{0, 1\}$; $\deg b_i = \deg c_j = \deg a_{ij} = \deg \delta = 1$.

$\mathcal{A}^{2Dv}$ is $\mathbb{Q}[\![\delta]\!]FA(b_i, c_j, a_{ij})$ (so $\mathcal{L}^v = \{f + f^{ij} a_{ij}\}$) modulo locality,

tt. (note $\gamma_{j\{kl\}} = 0$) $[a_{jk}, a_{jl}] = c_l a_{jk} - c_k a_{jl} =: \gamma_{jkl}$,

hh. $[a_{jk}, a_{ik}] = b_i a_{jk} - b_j a_{ik}$,

ht. $[a_{jk}, a_{kl}] = b_j a_{kl} - b_k a_{jl} - \gamma_{jkl}$,

ab,ac. $\operatorname{ad} a_{jk}\colon b_j, -b_k, -c_j, c_k \mapsto \delta a_{ij} - b_i c_j =: \gamma_{ij}$,

⇆. $[a_{ij}, a_{ji}] = ?$,

bc. $[b_i, c_j] = 0$.

So $a_{ij} f = f^\delta a_{ij} - \frac{b_i c_j}{\delta}(f^\delta - f)$, $[a_{ij}, f] = (f^\delta - f)\left(a_{ij} - \frac{b_i c_j}{\delta}\right)$,

with $f^\delta := f /\!/ \begin{pmatrix} b_i \to b_i + \delta & b_j \to b_j - \delta \\ c_i \to c_i - \delta & c_j \to c_j + \delta \end{pmatrix}$.

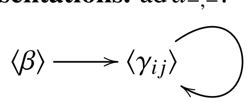
Repeating indices. In a_{ii}, γ_{ii}, γ_{iij}, γ_{ijj}, and γ_{iii}, heads above tails.

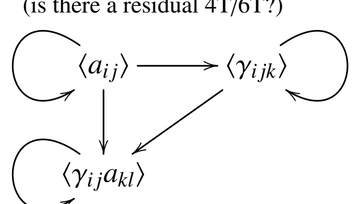
The Ascending Algebra $\mathcal{A}_+^{2Dv}$. Same but with only a_{ij}, $i < j$.

The OneCo Sub-Quotient is $\langle a_{ij}\rangle$ modulo $\delta^2 = \delta c_i = c_j c_k = 0$, so $\mathcal{L}^{1co}$ is (coefficient functions non-central, in $\mathbb{Q}[\![b_i]\!]$)

$$\left\{ f^{ij} a_{ij} + f^{ijk} \gamma_{ijk} + f^{ijkl} \gamma_{ij} a_{kl} \right\} \Bigg/ \begin{array}{l} r_1\colon\ b_i \gamma_{ijk} = \gamma_{ij} a_{ik} - \gamma_{ik} a_{ij} \\ r_2\colon\ b_i \gamma_{jkl} = \gamma_{jk} a_{il} - \gamma_{jl} a_{ik} \\ \text{(is there a residual 4T/6T?)} \end{array}.$$

Representations. $\operatorname{ad} a_{-,-}$:





• $\langle \gamma_{ij}\rangle$ is ...

OneCo $\operatorname{ad}(a_{jk})$: (authorities: pensieve://2015-06/)

$a[1,ij] \to a[-b_i,jk] + a[b_j,ik] + \gamma[1,ijk]$

$a[1,ik] \to a[b_i,jk] + a[-b_j,ik]$

$a[1,jl] \to \gamma[1,jkl]$

$a[1,kl] \to a[b_j,kl] + a[-b_k,jl] + \gamma[-1,jkl]$

$\beta[f[b_j, b_k]] \to \gamma[-f^{(0,1)}[b_j, b_k] + f^{(1,0)}[b_j, b_k], jk]$

$\gamma[1,ij] \to \gamma[b_j,ik]$

$\gamma[1,ik] \to \gamma[-b_j,ik]$

$\gamma[1,jk] \to \gamma[-b_j,jk]$

$\gamma[1,kl] \to \gamma[b_j,kl] + \gamma[-b_k,jl]$

$\gamma[1,ijk] \to \gamma[-b_j,ijk] + \gamma a[1,ijjk] + \gamma a[1,ikjk]$

$\gamma[1,ijl] \to \gamma a[1,ikjl]$

$\gamma[1,ikl] \to \gamma a[-1,ikjl]$

$\gamma[1,jkl] \to \gamma a[-1,jkjl]$

$\gamma[1,klm] \to \gamma[b_j,klm] + \gamma[-b_k,jlm]$

$\gamma a[1,jkjl] \to \gamma a[-b_j,jkjl]$

OneCo $\operatorname{Ad}(a_{jk})$:

$a[1,ij] \to a[1,ij] + a[1 - e^{-t b_j},ik] + a[\frac{(-1+e^{-t b_j}) b_i}{b_j},jk] +$
$\gamma[\frac{1-e^{-t b_j}}{b_j},ijk] + \gamma a[\frac{-1+e^{-t b_j}+t b_j}{b_j^2},ijjk] + \gamma a[\frac{-1+e^{-t b_j}+t b_j}{b_j^2},ikjk] +$
$\gamma a[\frac{b_i (1-e^{-2 t b_j}-2 e^{-t b_j} t b_j)}{b_j^3},jkjk] + \gamma a[\frac{e^{-2 t b_j} (1+e^{t b_j} (-1+t b_j))}{b_j^2},jkik]$

$a[1,ik] \to a[e^{-t b_j},ik] + a[\frac{(1-e^{-t b_j}) b_i}{b_j},jk] +$
$\gamma a[\frac{e^{-2 t b_j} b_i (1-e^{2 t b_j}+2 e^{t b_j} t b_j)}{b_j^3},jkjk] +$
$\gamma a[\frac{e^{-2 t b_j} (-1+e^{t b_j} (1-t b_j))}{b_j^2},jkik]$

$a[1,jl] \to a[1,jl] + \gamma[t,jkl] + \gamma a[\frac{1-e^{-t b_j}-t b_j}{b_j^2},jkjl]$

$a[1,kl] \to a[e^{t b_j},kl] + a[\frac{(1-e^{t b_j}) b_k}{b_j},jl] +$
$\gamma[\frac{t b_j b_k+(1-e^{t b_j}) (b_j+b_k)}{b_j^2},jkl] + \gamma a[\frac{1+e^{t b_j} (-1+t b_j)}{b_j^2},jkkl] +$
$\gamma a[\frac{-2 b_j+e^{-t b_j} b_j-2 b_k-t b_j b_k+e^{t b_j} (b_j+2 b_k-t b_j b_k)}{b_j^3},jkjl]$

$\beta[f[b_j, b_k]] \to$
$\gamma[\frac{(-1+e^{-t b_j}) (f^{(0,1)}[b_j,b_k]-f^{(1,0)}[b_j,b_k])}{b_j},jk] + \beta[f[b_j, b_k]]$

$\gamma[1,ij] \to \gamma[1,ij] + \gamma[1 - e^{-t b_j},ik]$

$\gamma[1,ik] \to \gamma[e^{-t b_j},ik]$

$\gamma[1,jk] \to \gamma[e^{-t b_j},jk]$

$\gamma[1,kl] \to \gamma[e^{t b_j},kl] + \gamma[\frac{(1-e^{t b_j}) b_k}{b_j},jl]$

$\gamma[1,ijk] \to \gamma[e^{-t b_j},ijk] + \gamma a[\frac{1-e^{-t b_j}}{b_j},ijjk] + \gamma a[\frac{1-e^{-t b_j}}{b_j},ikjk]$

$\gamma[1,ijl] \to \gamma[1,ijl] + \gamma[1 - e^{-t b_j},ikl] + \gamma a[\frac{1-e^{-t b_j}}{b_j},iljk]$

$\gamma[1,ikl] \to \gamma[e^{-t b_j},ikl] + \gamma a[\frac{-1+e^{-t b_j}}{b_j},iljk]$

$\gamma[1,jkl] \to \gamma[1,jkl] + \gamma a[\frac{-1+e^{-t b_j}}{b_j},jkjl]$

$\gamma[1,klm] \to \gamma[e^{t b_j},klm] + \gamma[\frac{(1-e^{t b_j}) b_k}{b_j},jlm]$

$\gamma a[1,jkjl] \to \gamma a[e^{-t b_j},jkjl]$

To do. • Perhaps I should find a way to highlight the fact that v is a perturbation of w. • Position FiC. • Position the 2D Lie bialgebras. • Is there a meaningful $a_{ij} \to a_{ij}/b_i$ (etc) renormalization? • Add: diagrammatic interpretations of b_i, c_j, γ_{ij}, γ_{ijk}.

Recycling.

Models. • In $[x,y] = \delta x$, $xf(y) = f(y+\delta)x$. If $\delta^2 = 0$, $[x, f(y)] = \delta f'(y)x$.

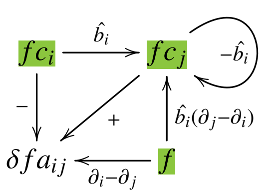
• In $[x,y] = \delta x + z^2$, $xf(y) = f(y+\delta)x + \frac{z^2}{\delta}(f(y+\delta) - f(y))$. If $\delta^2 = 0$, $[x, f(y)] = \delta f'(y)x + z^2 f'(y)$.

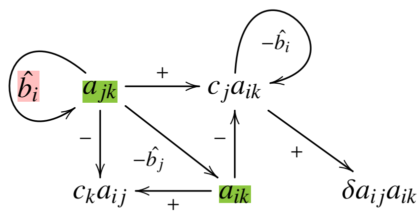
• If $S_n := \sum_{k=0}^{n-1} A^k C B^{n-1-k}$ then $AS_n - S_n B = A^n C - CB^n$ so $S_n = (L_A - R_B)^{-1}(A^n C - CB^n)$.

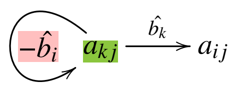
• If $\psi(x) = \sum_{n\geq 0} a_n x^n$ then $\sum_{n\geq 0} a_n \sum_{k=0}^{n-1} b^n(-b)^{n-1-k} = (\psi(b) - \psi(-b))/2b$.

The primitivity condition. $\ker\left(f + f^{ij}a_{ij} \mapsto \delta f + f^{ij} b_i c_j\right)$. (Ignoring multiple arrows).

State Diagrams. ad a_{ij} yields green: roots. pink: wrong.







so with $\phi_0 := \phi(0)$, $\phi_1 := \phi_0'$, and $\phi_\downarrow(x) := (\phi(x) - \phi_0)/x$, $\phi(\text{ad}\, a_{ij})$ is

$$\begin{aligned}
fc_i &\mapsto \phi_0 fc_i + (b_i\phi_{\downarrow\downarrow}(-b_i) - \phi_1)\delta f a_{ij} + b_i\phi_\downarrow(-b_i) fc_j \\
fc_j &\mapsto \phi(-b_i) fc_j + \phi_\downarrow(-b_i)\delta f a_{ij} \\
f &\mapsto \phi_0 f + b_i\phi_\downarrow(-b_i)(\partial_j f - \partial_i f)c_j \\
&\qquad + (b_i\phi_{\downarrow\downarrow}(-b_i) - \phi_1)(\partial_j f - \partial_i f)\delta a_{ij} \\
\delta a_{..} &\mapsto \text{as in Adjoint Gassner} \\
a_{ik} &\mapsto \phi_0 a_{ik} + \phi_1 c_k a_{ij} - \phi_\downarrow(-b_i)c_j a_{ik} - \phi_{\downarrow\downarrow}(-b_i)\delta a_{ij} a_{ik} \\
a_{jk} &\mapsto \phi(b_i)a_{jk} - (\phi_\downarrow(b_i) + b_j\phi_{\downarrow\downarrow}(b_i))c_k a_{ij} - b_j\phi_\downarrow(b_i)a_{ik} \\
&\qquad + \frac{\phi(b_i) - \phi(-b_i) + b_j(\phi_\downarrow(b_i) - \phi_\downarrow(-b_i))}{2b_i} c_j a_{ik} \\
&\qquad + \frac{\phi_\downarrow(b_i) - \phi_\downarrow(-b_i) + b_j(\phi_{\downarrow\downarrow}(b_i) - \phi_{\downarrow\downarrow}(-b_i))}{2b_i} \delta a_{ij} a_{ik} \\
a_{kj} &\mapsto \\
a_{ij} &\mapsto a_{ij}
\end{aligned}$$

Then $[a_{ij}, f] = (\partial_i f - \partial_j f)\gamma_{ij}$ and

γb. $[\gamma_{ij}, b_l] = 0$ and $[\gamma_{ijk}, b_l] = 0$ incl. $l \in \{i, j, k\}$,

ttγ. $[a_{jk}, \gamma_{jl}] = 0$,

hhγ. $[a_{jk}, \gamma_{ik}] = -b_j\gamma_{ik}$,

thγ. $[a_{jk}, \gamma_{ij}] = b_j\gamma_{ik}$,

htγ. $[a_{jk}, \gamma_{kl}] = b_j\gamma_{kl} - b_k\gamma_{jl}$,

ttγ_3. $[a_{jk}, \gamma_{jlm}] = 0$,

thγ_3. $[a_{jk}, \gamma_{ijl}] = b_j\gamma_{ikl} + \gamma_{il}a_{jk}$,

htγ_3. $[a_{jk}, \gamma_{klm}] = b_k\gamma_{jkl} + b_j\gamma_{klm}$,

hhγ_3. $[a_{jk}, \gamma_{nik}] = -b_j\gamma_{nik} + \gamma_{ni}a_{jk}$,

$[a_{jk}, \gamma_{jkl}] = -\gamma_{jk}a_{jl}$,

$[a_{jk}, \gamma_{ijk}] = -b_j\gamma_{ijk} + \gamma_{ij}a_{jk} + \gamma_{ik}a_{jk}$.