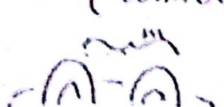
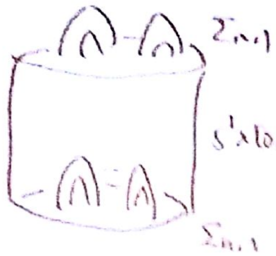
The category Cob (Crane-Yetter, Kerler)

$\mathrm{Ob}(\mathrm{Cob}) = \{0, 1, \cdots\}$

$\mathrm{Cob}(m, n) = \{$ connected, oriented 3d cobordisms $\Sigma_{m,1} \to \Sigma_{n,1} \} / \simeq$

$\Sigma_{m,1} =$  genus m

$M : m \to n$



$\partial M \cong -\Sigma_{m,1} \cup S^1\times[0,1] \cup \Sigma_{n,1}$

(Crane-Yetter, Kerler)

Fact (1) Cob is a braided monoidal category.

(2) $1 \in \mathrm{Ob}(\mathrm{Cob})$ admits a Hopf algebra (object) structure in Cob,

Fact Aut [illegible]

$\Sigma_{n,1}$

(Crane–Yetter, Kerler)

Fact (1) Cob is a braided monoidal category.

(2) $1 \in Ob(Cob)$ admits a Hopf algebra (object) structure in Cob,

Fact. $Aut_{Cob}(m) \cong \mathcal{M}_{m,1}$ the mapping class group of $\Sigma_{m,1}$.

Hennings invariant

H factorizable ribbon Hopf algebra over a field k.
with

$R \in H \otimes H$ the universal R-matrix

$v \in H$ the ribbon element $\rightsquigarrow$ $g = v^{-1}u$

$\lambda : H \to k$ a right integral $(\lambda(1) \neq 0)$.

$M^3 = S^3_L$

grouplike $(1, g$ or $g^{-1})$

multiplications $\rightsquigarrow$ element in $H \otimes H$

R-matrices

grouplike

$$\Big\downarrow \lambda(g\,-)^{\otimes 2} \qquad g \in H \text{ grouplike}$$

k

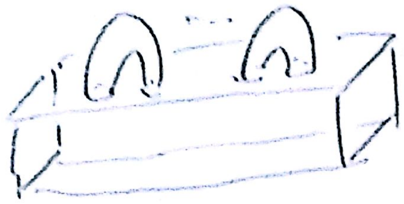
The element obtained in this way is a 3-manifold invariant after normalization.

The category $\mathcal{B}$ (1).

$Ob(\mathcal{B}) = \{0, 1, \cdots\}$

$\mathcal{B}(m, n) = \{n\text{-component bottom tangles in } V_m\} / \cong$

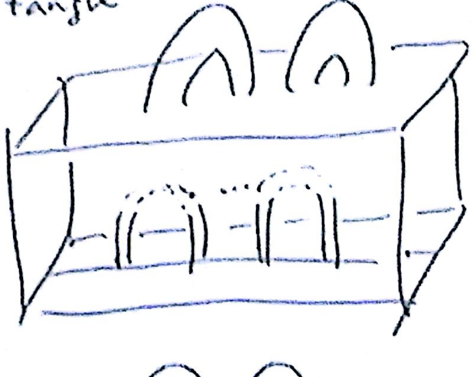
$V_m =$



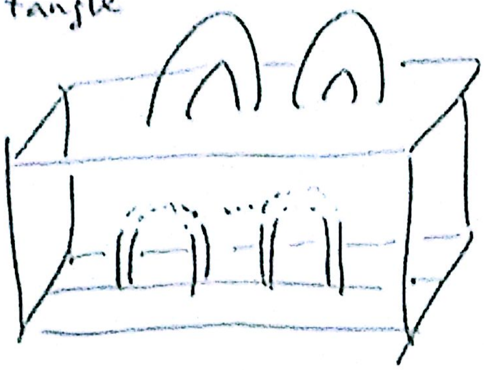
genus m handlebody

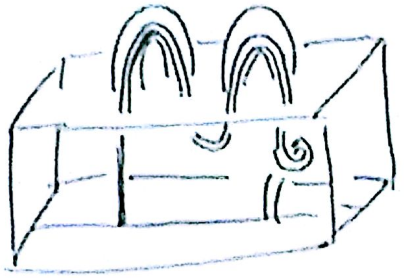
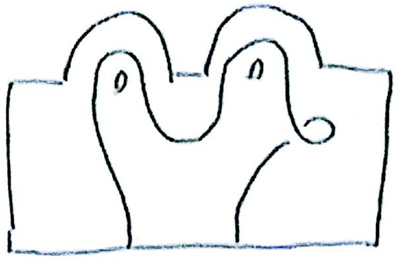
$= \text{cube} \cup (m \text{ 1-handles})$

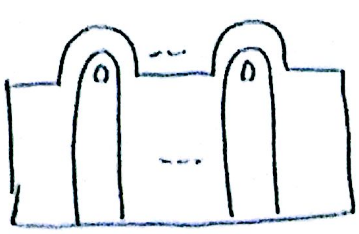
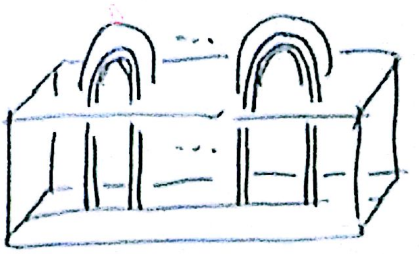
bottom tangle



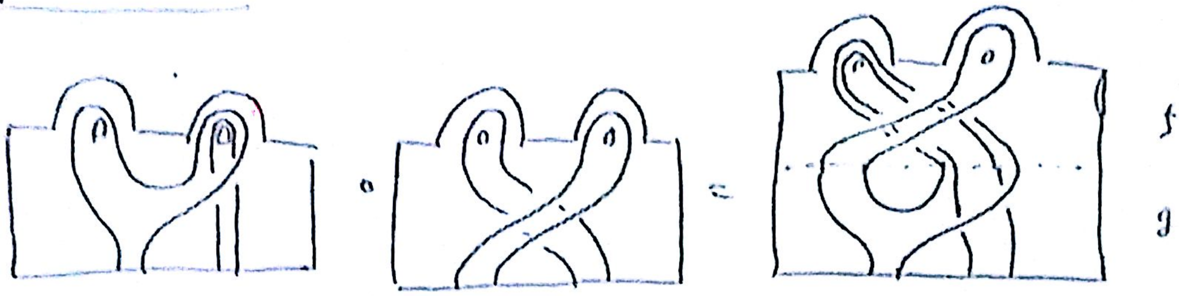
bottom tangle



e.g.  $\longleftrightarrow$ 

$1_m =$  $\Bigg(=$  $\Bigg) : m \to m$

Composition in $\mathcal{B}$



g $\qquad$ f $\qquad$ " g embedded along f "

<u>Remark</u> $\mathcal{B} \cong \mathcal{H}^{op}$, where $\mathcal{H}(m,n) = \{\text{embeddings } V_m \hookrightarrow V_n \text{ fixing bottom square}\}$ / isotopy

Hence $\mathrm{Aut}_{\mathcal{B}}(m) = \{\text{homeomorphisms } f : V_m \to V_m \text{ fixing bottom square}\}/\text{isotopy}.$

$\subset \mathcal{M}_{m,1}.$

$\otimes$ on $\mathcal{B}$.

$m \otimes n = m+n$

$\otimes$ on $\mathcal{B}$.

$m \otimes n = m + n$.

monoidal unit $= 0$.

braiding

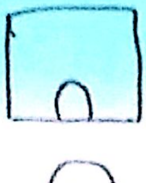
$\Psi = \Psi_{1,1} : \underset{\substack{\| \\ 2}}{1 \otimes 1} \longrightarrow \underset{\substack{\| \\ 2}}{1 \otimes 1}$

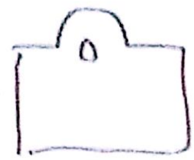
$\Psi =$ 

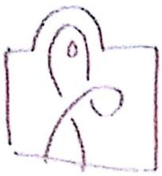
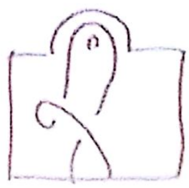
$\Psi^{-1} =$ 

$\rightsquigarrow$ $\mathcal{B}$ is a braided monoidal category

Hopf algebra in B

$\mu =$  $=$  , $\eta =$  $=$ 

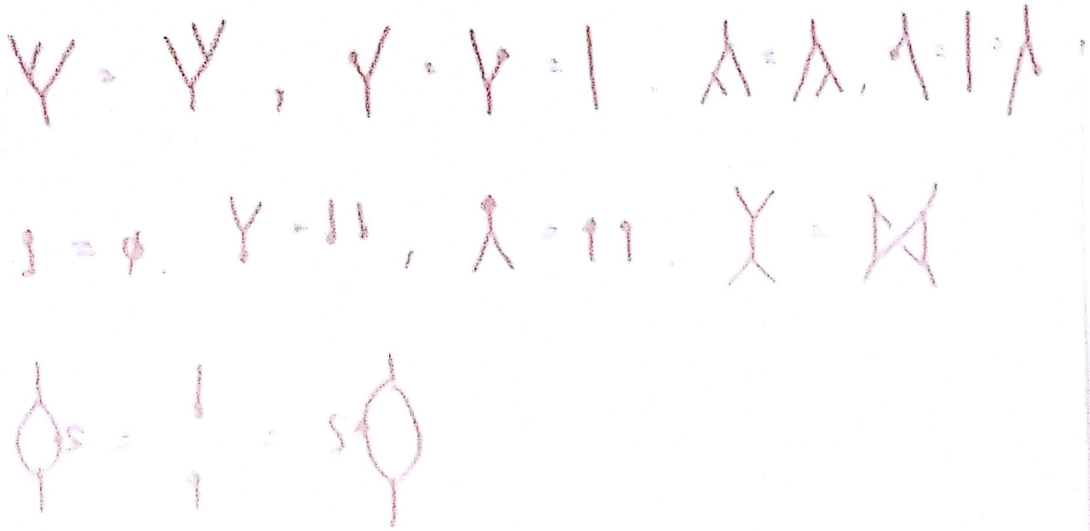
$\Delta =$  $=$  , $\varepsilon =$  $=$ 

$S =$  $=$  , $S^{-1} =$  $=$ 

. Hopf algebra relations

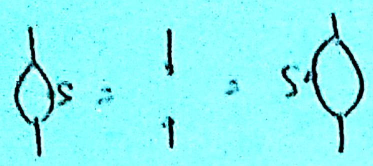
(4-1)

Hopf algebra relations

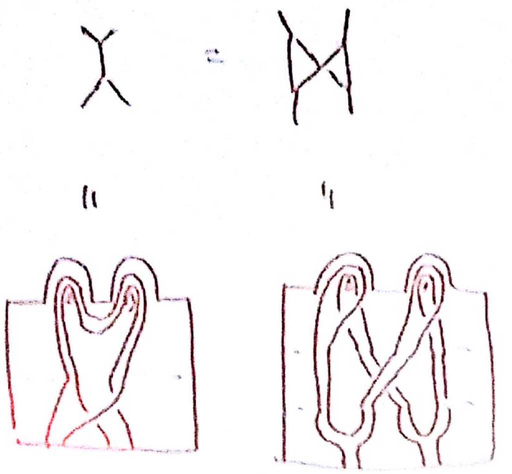


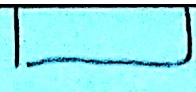
$0 \longrightarrow 1.$

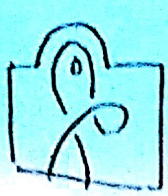
0 → 1.



6-1



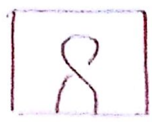
$\Delta =$  $=$  , $\varepsilon =$ 

$S =$  $=$  , $S^{-1} =$  $=$ 

Fact (H)

$\mathcal{B}$ is generated as a monoidal category by

$\psi, \psi^{-1}, \mu, \eta, \Delta, \varepsilon, S, S^{-1}$, and

$r^{+} =$  , $r^{-} =$  $: 0 \to 1$.

(ribbon element)

I Half self duality in $\mathcal{B}$ (cf. self duality in Cob (Kerler))

$c = \bigcap =$  $: 0 \longrightarrow 2$. "clasp element".

We have

The functor $J : \mathcal{B} \longrightarrow \mathrm{Mod}_H$

⑬

· related to the universal quantum invariants of links and tangles.
Hennings, Lawrence, Reshetikhin, Ohtsuki, Kauffman-Radford, Kerler

Let H be a ribbon Hopf algebra with $R = \sum \alpha \otimes \beta$

The monoidal functor $J : \mathcal{B} \longrightarrow \mathrm{Mod}_H$ is characterized by

$J(n) = \underline{H}^{\otimes n}$,

$\underline{H} = (H, \mathrm{ad})$, $\mathrm{ad} : H \otimes H \longrightarrow H$
adjoint action

$\mathrm{ad}(x \otimes y) := \sum x_{(1)} y S(x_{(2)})$
$\parallel$
$x \triangleright y$.
Here $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$

$J(\psi^{\pm 1}) = \psi^{\pm 1}_{\underline{H},\underline{H}}$, $\psi_{\underline{H},\underline{H}}(x \otimes y) = \sum \beta \triangleright y \otimes \alpha \triangleright x$

$J(\mu) = \mu_H$, $\mu(x \otimes y) = xy$

$\underline{H} = (H, \mathrm{ad})$. $\quad \mathrm{ad} : H \otimes H \longrightarrow H$
adjoint action

$\mathrm{ad}(x \otimes y) := \sum x_{(1)} y S(x_{(2)})$
$\parallel$
$x \triangleright y$.

Here $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

$J(\Psi^{\pm 1}) = \Psi^{\pm 1}_{\underline{H},\underline{H}}$, $\quad \Psi_{\underline{H},\underline{H}}(x \otimes y) = \sum \beta \triangleright y \otimes \alpha \triangleright x$

$J(\mu) = \mu_H$, $\quad \mu(x \otimes y) = xy$.

$J(\eta) = \eta_H : k \longrightarrow H$.

$J(\Delta) = \underline{\Delta}$, $\quad \underline{\Delta}(x) := \sum x_{(1)} S(\beta) \otimes \alpha \triangleright x_{(2)}$

(Majid's twisted comultiplication)

$J(\varepsilon) = \varepsilon_H$

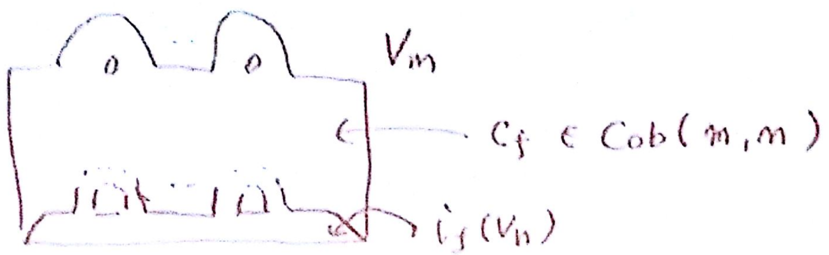
$J(S^{\pm 1}) = \underline{S}^{\pm 1}$, $\quad \underline{S}(x) = \sum \beta S(\alpha \triangleright x)$

$J(r^{\pm}) = v^{\pm 1} : k \longrightarrow H$

$\mathcal{B}$ as a subcategory of Cob

For $f: m \longrightarrow n \rightsquigarrow i_f: V_m \hookrightarrow V_n$.

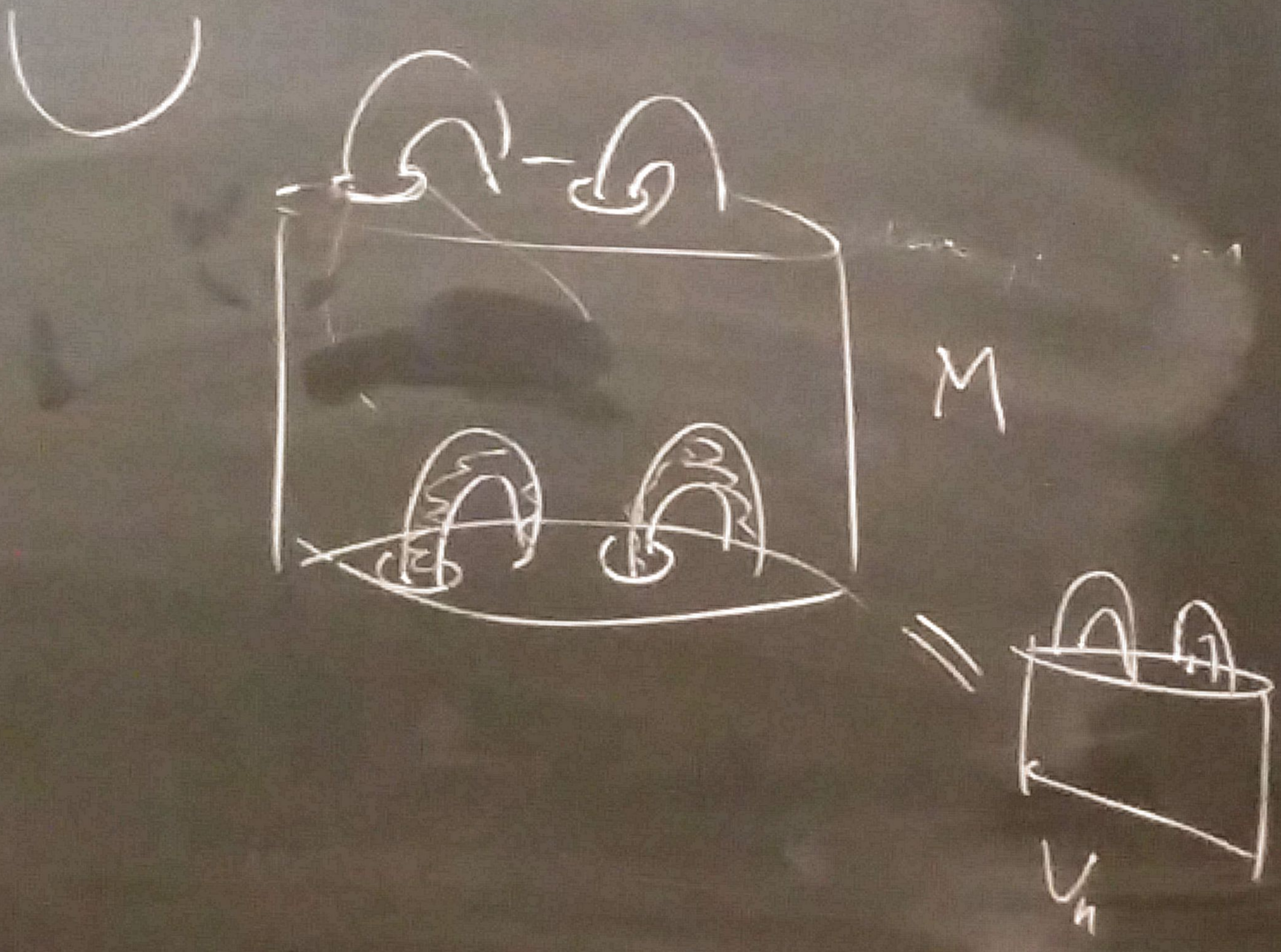
$C_f := V_m \setminus i_f(V_n)$, where we choose i_f so that $i_f(V_n) \cap \partial V_m = $ (bottom square of V_m)



We have a (faithful) functor

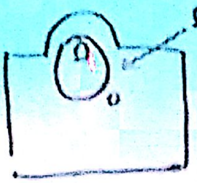
$$C: \mathcal{B} \longrightarrow \mathrm{Cob}$$
$$f \longmapsto C_f$$

Regard $\mathcal{B}$ as a subcategory of Cob.



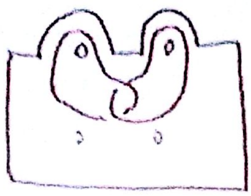
$f_*(A)=A$

Morphisms in Cob, not in $\mathcal{B}$

$\lambda :=$ 

$: 1 \to 0$ an integral in Cob

$d :=$ 

$:=$  $:$  $=$ 

$\|$

Self-Duality in Cob

 $= | =$  in Cob

$| =$ 

Hennings TQFT as extension of $J : \mathcal{B} \to \mathrm{Mod}_H$

H : factorizable (fin. dim.) ribbon Hopf algebra over a field k.

H factorizable $\overset{\text{def}}{\iff}$ $c_H := J(c) = (S \otimes 1)(R_{21}R) \in H \otimes H$ is nondegenerate

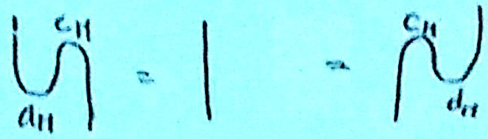
i.e., there is $d_H : H \otimes H \to k$ such that

<u>Fact</u> Cob is generated as a monoidal category by the morphisms in

and either λ or d.

<u>Fact.</u> The Hennings TQFT is the unique extension of $J : \mathcal{B} \to \mathrm{Mod}_H$

s.t. $Z(d) = d_H$.



<u>Fact</u> Cob is generated as a monoidal category by the morphisms in B and either λ or d.

<u>Fact.</u> The Hennings TQFT is the unique extension of $J : B \to Mod_H$ s.t. $Z(d) = d_H$.

$$\begin{tikzcd} B \arrow[dr, "J"] \arrow[d, phantom, "\cap"] & \\ Cob \arrow[r, "Z"'] & Mod_H \end{tikzcd}$$

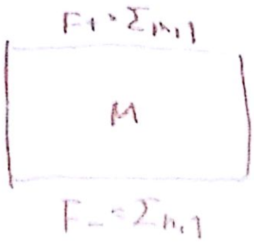
The category $\mathcal{L}Cob$. (Cheptea - H - Massuyeau)

$\mathcal{L}Cob$ is a subcategory of Cob such that

$$Ob(\mathcal{L}Cob) = \{0, 1, \cdots\}$$

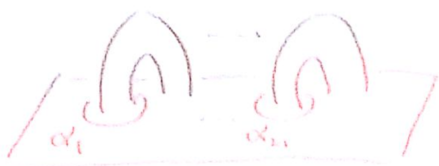
$$\mathcal{L}Cob(m,n) = \left\{ f = [M] : m \to n \text{ in } Cob \text{ such that } \begin{array}{l} (1)\ H_1(M;\mathbb{Z}) = i_*^-(A_-) + i_*^+(H_1(F_+;\mathbb{Z})) \\ (2)\ i_*^+(A_+) \subset i_*^-(A_-). \end{array} \right\}$$

Here



$$\begin{array}{c} F_+ \\ \downarrow i^+ \\ M \\ \uparrow i^- \\ F_- \end{array}$$

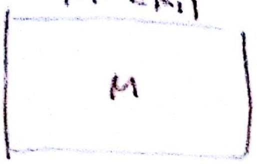
$A_+ = \mathbb{Z}\{\alpha_1, \cdots, \alpha_m\}$



$$\mathcal{L}Cob(m,n) = \left\{ \begin{array}{l} f=[M]: m \to n \text{ in } Cob \text{ such that} \\ (1)\ H_1(M;\mathbb{Z}) = i_*^-(A_-) + i_*^+(H_1(F_+;\mathbb{Z})) \\ (2)\ i_*^+(A_+) \subset i_*^-(A_-). \end{array} \right\}$$

Here

$F_+ = \Sigma_{m,1}$



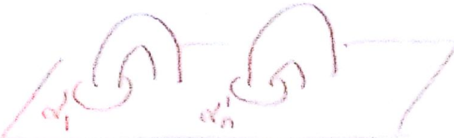
$F_- = \Sigma_{n,1}$

$$\begin{array}{c} F_+ \\ \downarrow i^+ \\ M \\ \uparrow i^- \\ F_- \end{array}$$

$A_+ = \mathbb{Z}\{\alpha_1^+, \ldots, \alpha_m^+\}$

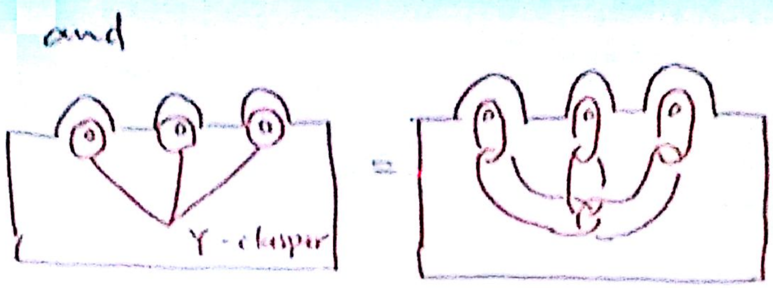
(Lagrangian submodule of $H_1 F_+$)

$A_- = \mathbb{Z}\{\alpha_1^-, \ldots, \alpha_n^-\}$



Fact (Cheptea-H-Massuyeau)

$\mathcal{L}Cob$ is generated as a monoidal category by the morphisms in $\mathcal{B}$ and

$Y_3 = \Psi :=$ 

$: 3 \to 0$

How to extend $J : \mathcal{B} \to \mathrm{Mod}_H$ to $\mathcal{L}Cob$

$\rightsquigarrow$ Define $J(Y_3) : H^{\otimes 3} \to k$.

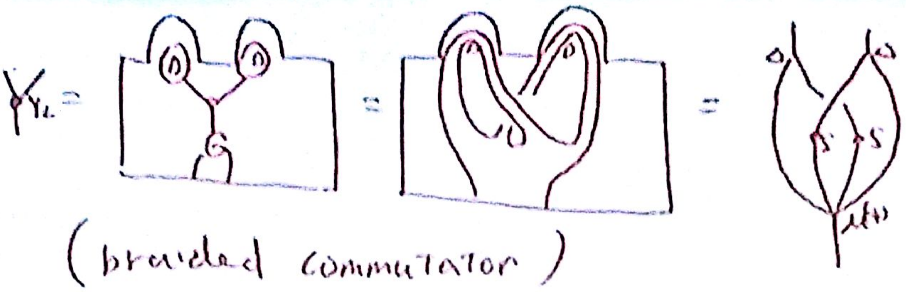
If H is factorizable, then it suffices to set

$$J(Y_3) = d \circ (J(Y_2) \otimes \mathrm{id}_H).$$

If H is factorizable, then it suffices to set

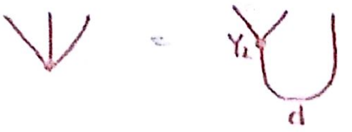
$$J(Y_3) = d \circ (J(Y_2) \otimes \mathrm{id}_H).$$

Here

$Y_2 =$  $: 2 \to 1$

(braided commutator)

We have

$Y_3 =$  in Cob.

In the quantum group case $H = U_h(\mathfrak{g})$, we can define d as a partial map

$$d_H : U_h \hat{\otimes} U_h \dashrightarrow k = \mathbb{Q}[[h]]$$

such that

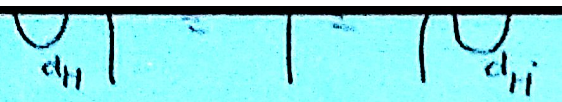
$: U_h \longrightarrow U_h$

Then, define a partial map

$J(Y_3) :=$ 

$: U_h^{\hat{\otimes} 3} \longrightarrow \mathbb{Q}[[h]].$

This works to define a functor



$: U_h \longrightarrow U_h$

Then, define a partial map

$J(Y_3) :=$ 

$: U_h^{\hat{\otimes} 3} \longrightarrow \mathbb{Q}[[h]].$

This works to define a functor

$J : \mathcal{L}Cob \longrightarrow Mod_{\mathbb{Q}[[h]]}$ (at least in the sl_2 case)

Remaining thing to do is to construct a subfunctor

$J : \mathcal{L}Cob \longrightarrow Mod_{\mathbb{Z}[q]}.$