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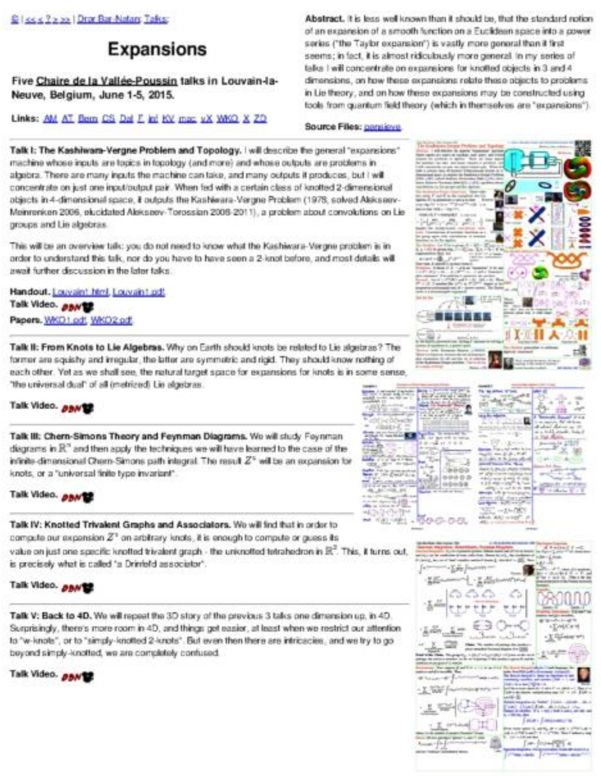

© | << < ? > >> | Dror Bar-Natan: Talks:

Expansions

Five Chaire de la Vallée-Poussin talks in Louvain-la-Neuve, Belgium, June 1-5, 2015.

Links: AM AT Bern CS Dal F inf KV mac uX WKO X ZD

Abstract. It is less well known than it should be, that the standard notion of an expansion of a smooth function on a Euclidean space into a power series ("the Taylor expansion") is vastly more general than it first seems; in fact, it is almost ridiculously more general. In my series of talks I will concentrate on expansions for knotted objects in 3 and 4 dimensions, on how these expansions relate these objects to problems in Lie theory, and on how these expansions may be constructed using tools from quantum field theory (which in themselves are "expansions").

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Talk I: The Kashiwara-Vergne Problem and Topology. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978, solved Alekseev-Meinrenken 2006, elucidated Alekseev-Torossian 2008-2011), a problem about convolutions on Lie groups and Lie algebras.

This will be an overview talk; you do not need to know what the Kashiwara-Vergne problem is in order to understand this talk, nor do you have to have seen a 2-knot before, and most details will await further discussion in the later talks.

Handout. Louvain1.html, Louvain1.pdf
Talk Video.
Papers. WKO1.pdf, WKO2.pdf

Talk II: From Knots to Lie Algebras. Why on Earth should knots be related to Lie algebras? The former are squishy and irregular, the latter are symmetric and rigid. They should know nothing of each other. Yet as we shall see, the natural target space for expansions for knots is in some sense, "the universal dual" of all (metrized) Lie algebras.

Talk Video.

Talk III: Chern-Simons Theory and Feynman Diagrams. We will study Feynman diagrams in $\mathbb{R}^n$ and then apply the techniques we will have learned to the case of the infinite-dimensional Chern-Simons path integral. The result Z^κ will be an expansion for knots, or a "universal finite type invariant".

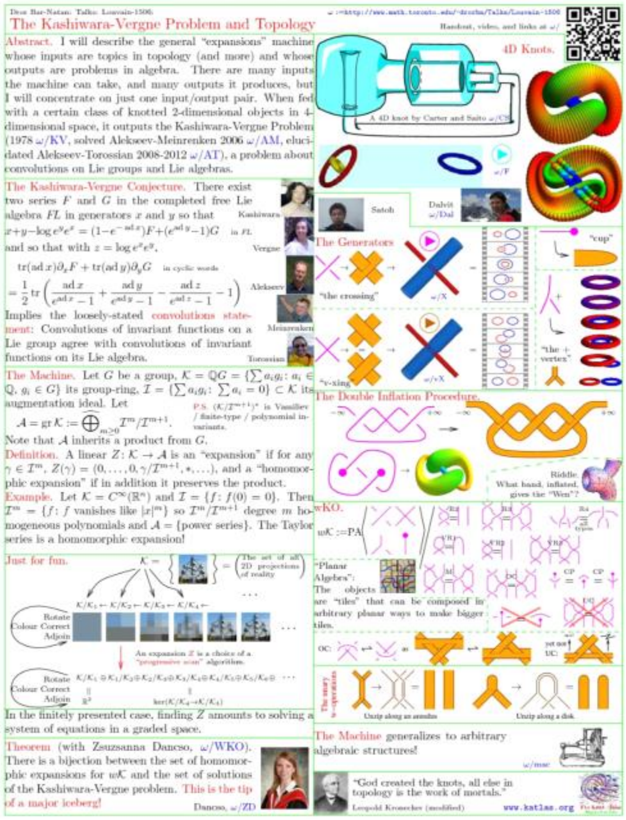
Talk Video.

Talk IV: Knotted Trivalent Graphs and Associators. We will find that in order to compute our expansion Z^κ on arbitrary knots, it is enough to compute or guess its value on just one specific knotted trivalent graph - the unknotted tetrahedron in $\mathbb{R}^3$. This, it turns out, is precisely what is called "a Drinfeld associator".

Talk Video.

Talk V: Back to 4D. We will repeat the 3D story of the previous 3 talks one dimension up, in 4D. Surprisingly, there's more room in 4D, and things get easier, at least when we restrict our attention to "w-knots", or to "simply-knotted 2-knots". But even then there are intricacies, and we try to go beyond simply-knotted, we are completely confused.

Talk Video.



Dror Bar-Natan: Talks: Louvain-1506.

The Kashiwara-Vergne Problem and Topology

Abstract. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 ω/KV, solved Alekseev-Meinrenken 2006 ω/AM, elucidated Alekseev-Torossian 2008-2012 ω/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra FL in generators x and y so that
$x+y-\log e^y e^x = (1-e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y}-1)G$ in FL
and so that with $z=\log e^x e^y$,
$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G$ in cyclic words
$= \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1\right)$
Implies the loosely-stated convolutions statement: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.

The Machine. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i : a_i \in \mathbb{Q}, g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i : \sum a_i = 0\} \subset \mathcal{K}$ its augmentation ideal. Let
$\mathcal{A} = \operatorname{gr}\mathcal{K} := \bigoplus_{m \geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}$.
Note that $\mathcal{A}$ inherits a product from G.
Definition. A linear $Z: \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0,\ldots,0,\gamma/\mathcal{I}^{m+1},\ast,\ldots)$, and a "homomorphic expansion" if in addition it preserves the product.
Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f : f(0)=0\}$. Then $\mathcal{I}^m = \{f : f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun.

In the finitely presented case, finding Z amounts to solving a system of equations in a graded space.

Theorem (with Zsuzsanna Dancso, ω/WKO). There is a bijection between the set of homomorphic expansions for $w\mathcal{K}$ and the set of solutions of the Kashiwara-Vergne problem. This is the tip of a major iceberg!

The Generators. The Double Inflation Procedure. The Machine generalizes to arbitrary algebraic structures!

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

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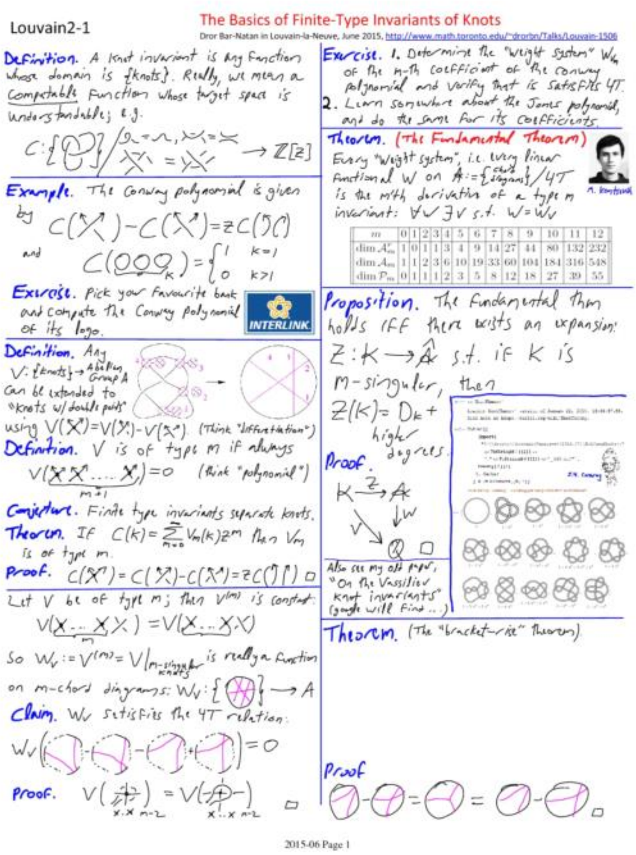

Louvain2-1

The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

Example. The Conway polynomial is given by $C(\times) - C(\times) = zC()()$ and $C(\bigcirc\bigcirc\bigcirc_k) = \begin{cases}1 & k=1\\ 0 & k>1\end{cases}$

Exercise. Pick your favourite knot and compute the Conway polynomial of its logo. INTERLINK

Definition. Any V: {knots} → Abelian Group A can be extended to "knots w/ double points" using $V(\times) = V(\times) - V(\times)$. (Think "differentiation")

Definition. V is of type m if always $V(\times\times\ldots\times_{m+1}) = 0$ (think "polynomial")

Conjecture. Finite type invariants separate knots.

Theorem. If $C(k) = \sum_{m=0}^{\infty} V_m(k) z^m$ then V_m is of type m.

Proof. $C(\times) = C(\times) - C(\times) = zC()() \ldots$

Let V be of type m; then $V^{(m)}$ is constant: $V(\times\ldots\times\times) = V(\times\ldots\times\times)$

So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V: \{\ldots\} \to A$

Claim. W_V satisfies the 4T relation: $W_V(\ldots - \ldots - \ldots + \ldots) = 0$

Proof. $V(\ldots) = V(\ldots)$

Exercise. 1. Determine the "weight system" W_V of the n-th coefficient of the Conway polynomial and verify that it satisfies 4T. 2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m-th derivative of a type m invariant: $\forall W\, \exists V$ s.t. $W = W_V$. M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	181	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Proposition. The Fundamental theorem holds iff there exists an expansion $Z: \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K + \text{higher degrees}$

Proof.

Also see my old paper, "On the Vassiliev knot invariants" (google will find...)

Theorem. (The "bracket-rise" theorem)

Proof

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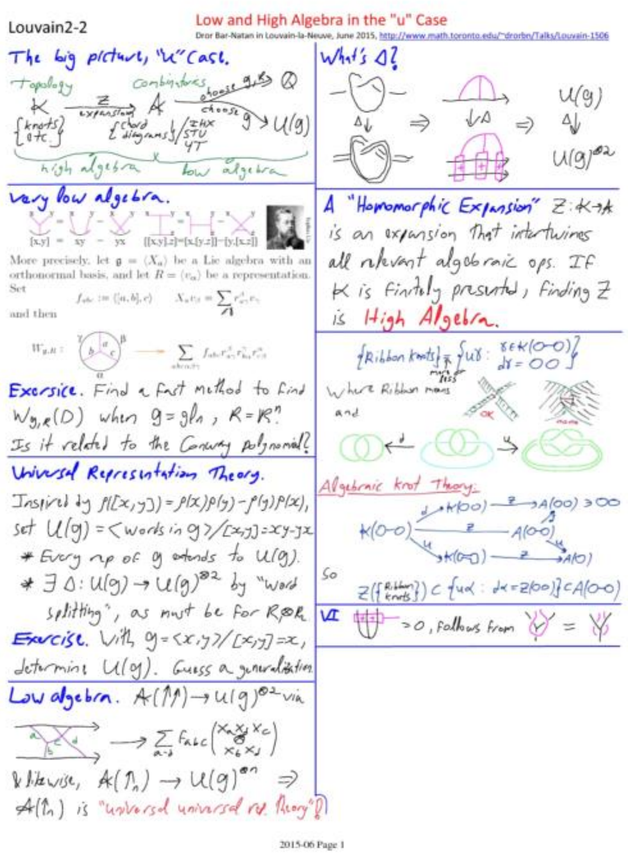

Louvain2-2

Low and High Algebra in the "u" Case

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

The big picture, "u" case.

Topology — Combinatorics — choose $\mathfrak{g}, R$ — $\mathbb{Q}$

$\mathcal{K}$ {knots etc.} $\xrightarrow[\text{expansion}]{Z} \mathcal{A}$ {chord diagrams}/STU 4T $\xrightarrow{\text{choose } \mathfrak{g}} \mathcal{U}(\mathfrak{g})$

high algebra — low algebra

Very low algebra.

$[x,y] = xy - yx$ $\quad [[x,y],z] = [x,[y,z]] - [y,[x,z]]$

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = (v_\alpha)$ be a representation. Set $f_{abc} := \langle [a,b], c\rangle$, $X_a v_\beta = \sum_\gamma r^\gamma_{a\beta} v_\gamma$ and then

$W_{\mathfrak{g},R}: \ldots \longrightarrow \sum f_{abc} r^{\ldots} r^{\ldots} r^{\ldots}$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$. Is it related to the Conway polynomial?

Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal{U}(\mathfrak{g}) = \langle \text{words in } \mathfrak{g}\rangle / ([x,y] = xy - yx)$

∗ Every rep of $\mathfrak{g}$ extends to $\mathcal{U}(\mathfrak{g})$.

∗ $\exists\, \Delta: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R \otimes R$.

Exercise. With $\mathfrak{g} = \langle x,y\rangle/([x,y] = x)$, determine $\mathcal{U}(\mathfrak{g})$. Guess a generalization.

Low algebra. $\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via

$\ldots \longrightarrow \sum_{a,b} f_{abc} (X_a X_b X_c)$

Likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n}$ ⇒ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

$\Delta\downarrow \Rightarrow \downarrow\Delta \Rightarrow$ $\mathcal{U}(\mathfrak{g})$ $\Delta\downarrow$ $\mathcal{U}(\mathfrak{g})^{\otimes 2}$

A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$ is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is High Algebra.

{Ribbon knots} $\xrightarrow{Z}$ {$u\delta$: $\delta\epsilon\mathcal{K}(\bigcirc\!\!-\!\!\bigcirc)$, $d\gamma = \bigcirc\bigcirc$}, where Ribbon means, and

Algebraic knot theory.

$\mathcal{K}(\bigcirc\!-\!\bigcirc) \to \mathcal{A}(\bigcirc\!-\!\bigcirc)$

So $Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\} \subset \mathcal{A}(\bigcirc\!-\!\bigcirc)$

VI $\ldots \geq 0$, follows from $\ldots = \ldots$

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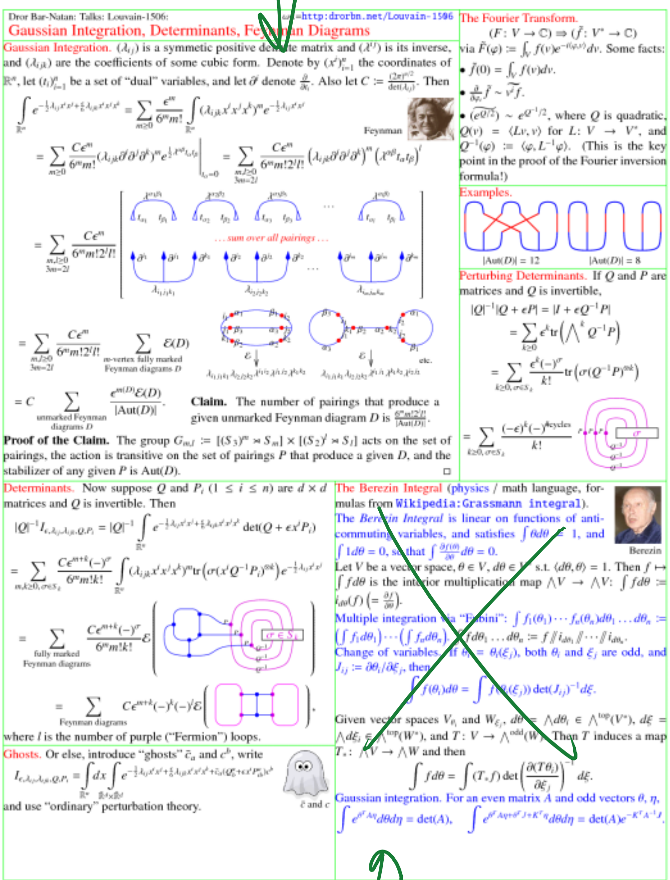
UFTI stuff



Dror Bar-Natan: Talks: Louvain-1506: ω:=http:drorbn.net/Louvain-1506

Gaussian Integration, Determinants, Feynman Diagrams

Gaussian Integration. (λ_{ij}) is a symmetic positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$

Feynman

$$= \sum_{m\geq 0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\bigg|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left|\ \cdots\ \right|$$

... sum over all pairings ...

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$

etc.

$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m] \times [(S_2)^l \rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

The Fourier Transform.

$$(F: V\to\mathbb{C}) \Rightarrow (\tilde{f}: V^*\to\mathbb{C})$$

via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.

$|\mathrm{Aut}(D)| = 12$ $\quad$ $|\mathrm{Aut}(D)| = 8$

Perturbing Determinants. If Q and P are matrices and Q is invertible,

$$|Q|^{-1}|Q+\epsilon P| = |I+\epsilon Q^{-1}P| = \sum_{k\geq 0}\epsilon^k\mathrm{tr}\left(\bigwedge^k Q^{-1}P\right) = \sum_{k\geq 0,\,\sigma\in S_k}\frac{\epsilon^k(-)^\sigma}{k!}\mathrm{tr}\left(\sigma(Q^{-1}P)^{\otimes k}\right)$$

$$= \sum_{k\geq 0,\,\sigma\in S_k}\frac{(-\epsilon)^k(-)^{\#\text{cycles}}}{k!}$$

Determinants. Now suppose Q and P_i $(1\leq i\leq n)$ are $d\times d$ matrices and Q is invertible. Then

$$|Q|^{-1}I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = |Q|^{-1}\int_{\mathbb{R}^n}e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}\det(Q+\epsilon x^iP_i)$$

$$= \sum_{m,k\geq 0,\,\sigma\in S_k}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m\mathrm{tr}\left(\sigma(x^iQ^{-1}P_i)^{\otimes k}\right)e^{-\frac{1}{2}\lambda_{ij}x^ix^j}$$

$$= \sum_{\substack{\text{fully marked}\\ \text{Feynman diagrams}}}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\mathcal{E}\left(\ \cdots\ \right)$$

$$= \sum_{\text{Feynman diagrams}}C\epsilon^{m+k}(-)^k(-)^l\mathcal{E}\left(\ \cdots\ \right),$$

where l is the number of purple ("Fermion") loops.

Ghosts. Or else, introduce "ghosts" $\bar{c}_a$ and c^b, write

$$I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = \int_{\mathbb{R}^n}dx\int_{\mathbb{R}^{d}\times\mathbb{R}^{d}}e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k+\bar{c}_a(Q^a_b+\epsilon x^iP^a_{ib})c^b}$$

and use "ordinary" perturbation theory.

$\bar{c}$ and c

The Berezin Integral (physics / math language, formulas from Wikipedia:Grassmann integral).

Berezin

The *Berezin Integral* is linear on functions of anti-commuting variables, and satisfies $\int\theta d\theta = 1$, and $\int 1d\theta = 0$, so that $\int\frac{\partial f(\theta)}{\partial\theta}d\theta = 0$.

Let V be a vector space, $\theta\in V$, $d\theta\in V^*$ s.t. $\langle d\theta,\theta\rangle = 1$. Then $f\mapsto\int fd\theta$ is the interior multiplication map $\bigwedge V\to\bigwedge V$: $\int fd\theta := i_{d\theta}(f)\left(=\frac{\partial f}{\partial\theta}\right)$.

Multiple integration via "Fubini": $\int f_1(\theta_1)\cdots f_n(\theta_n)d\theta_1\ldots d\theta_n := \left(\int f_1d\theta_1\right)\cdots\left(\int f_nd\theta_n\right)$, $\int fd\theta_1\ldots d\theta_n := f /\!\!/ i_{d\theta_1} /\!\!/ \cdots /\!\!/ i_{d\theta_n}$.

Change of variables. If $\theta_i = \theta_i(\xi_j)$, both θ_i and ξ_j are odd, and $J_{ij} := \partial\theta_i/\partial\xi_j$, then

$$\int f(\theta_i)d\theta = \int f(\theta_i(\xi_j))\det(J_{ij})^{-1}d\xi.$$

Given vector spaces V_{θ_i} and W_{ξ_j}, $d\theta = \bigwedge d\theta_i\in\bigwedge^{\mathrm{top}}(V^*)$, $d\xi = \bigwedge d\xi_j\in\bigwedge^{\mathrm{top}}(W^*)$, and $T\colon V\to\bigwedge^{\mathrm{odd}}(W)$. Then T induces a map $T_*\colon\bigwedge V\to\bigwedge W$ and then

$$\int fd\theta = \int(T_*f)\det\left(\frac{\partial(T\theta_i)}{\partial\xi_j}\right)^{-1}d\xi.$$

Gaussian integration. For an even matrix A and odd vectors θ, η,

$$\int e^{\theta^TA\eta}d\theta d\eta = \det(A),\qquad \int e^{\theta^TA\eta+\theta^TJ+K^T\eta}d\theta d\eta = \det(A)e^{-K^TA^{-1}J}.$$

reconsider.

CFI stuff, chopsticks.