

Louvain planning

May-15-15 8:31 PM

✓ Ask about colour printer in Louvain.
✓ Make a 2-page montage.

Dror Bar-Natan: Talks: Louvain-1306

1 of 3

© | <<< ? >>> | [Dror Bar-Natan: Talks:](#)

Expansions

Five Chaire de la Vallée-Poussin talks in Louvain-la-Neuve, Belgium, June 1-5, 2015.

Links: [AM](#) [AT](#) [Bern](#) [CS](#) [Dal](#) [F](#) [inf](#) [KV](#) [mac](#) [vX](#) [WKO](#) [X](#) [ZD](#)

Abstract. It is less well known than it should be, that the standard notion of an expansion of a smooth function on a Euclidean space into a power series ("the Taylor expansion") is vastly more general than it first seems; in fact, it is almost ridiculously more general. In my series of talks I will concentrate on expansions for knotted objects in 3 and 4 dimensions, on how these expansions relate these objects to problems in Lie theory, and on how these expansions may be constructed using tools from quantum field theory (which in themselves are "expansions").

Source Files: [pensieve](#).

Talk I: The Kashiwara-Vergne Problem and Topology. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978, solved Alekseev-Meinrenken 2006, elucidated Alekseev-Torossian 2008-2011), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Problem and Topology
Abstract. I will describe the general "expansions" machine whose inputs are topics in topology (and more) and whose outputs are problems in algebra. There are many inputs the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978, solved Alekseev-Meinrenken 2006, elucidated Alekseev-Torossian 2008-2011), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra \widehat{FL} in generators x and y so that $x \cdot y - \log e^{x \cdot y} = (1 - e^{-ad_x})F + (e^{ad_x} - 1)G = 0$, and so that with $z = \log e^{x \cdot y}$, $\text{tr}(\text{ad } z)_0, F + \text{tr}(\text{ad } y)_0, G = 0$ in each weight $n \geq 1$.

The Machine. Let G be a group, $K = \mathbb{Q}\langle G \rangle = \sum_{n \geq 0} \mathfrak{L}_n$, $\mathfrak{L}_0 = \mathbb{Q}$, $\mathfrak{L}_1 = \mathfrak{g}$, its group-ring, $\mathbb{Z} = \sum_{n \geq 0} \mathfrak{L}_n$, $\mathfrak{L}_n = 0$ if $n < 0$. Its augmentation ideal, $\mathbb{Z}^+ = \sum_{n \geq 1} \mathfrak{L}_n$. Let $\mathfrak{L} = \mathfrak{L}^+ / \mathfrak{L}^+ \cdot \mathfrak{L}^+$ be the Lie algebra of Lie-type / polynomial invariants. Note that \mathfrak{L} inherits a product from G .

Definition. A linear $Z: K \rightarrow \mathfrak{L}$ is an "expansion" if for any $\gamma \in \mathbb{Z}^+, Z(\gamma) = (0, \dots, 0, \gamma / \mathbb{Z}^+, \dots)$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $K = C^\infty(\mathbb{R}^2)$ and $\mathbb{Z} = \{f: f(0) = 0\}$. Then $\mathbb{Z}^+ = \{f: f \text{ vanishes like } |x|^m\}$ as $\mathbb{Z}^+ / \mathbb{Z}^+ \cdot \mathbb{Z}^+$ degree or homogeneous polynomials and $\mathfrak{L} = \{\text{power series}\}$. The Taylor series is a homomorphic expansion!

Just for fun. $K = \mathfrak{L} = \left(\frac{\text{set of all } \mathbb{Z}^+ \text{ monomials}}{\text{if monomial}} \right)$

The Machine generalizes to arbitrary algebraic structures!

"God created the knots, all else in topology is the work of mortals."
 Leopold Kronecker (attributed) [see knotlib.org](#)

This will be an overview talk: you do not need to know what the Kashiwara-Vergne problem is in order to understand this

talk, nor do you have to have seen a 2-knot before, and most details will await further discussion in the later talks.

Handout. [Louvain1.html](#), [Louvain1.pdf](#).


Talk Video. 

Papers. [WKO1.pdf](#), [WKO2.pdf](#).

squishy and irreducible are symmetric & rigid

Handout based on Columbia LITS 3k! 2011-07

Talk II: From Knots to Lie Algebras. Why on Earth should knots be related to Lie algebras? The former are ~~about deformability~~, the latter ~~about symmetry and rigidity~~. They should know nothing of each other. Yet as we shall see, the natural target space for expansions for knots is in some sense, "the universal dual" of all (metrized) Lie algebras.

Talk Video. 

add some UFTI stuff.

Talk III: Chern-Simons Theory and Feynman Diagrams. We will study Feynman diagrams in \mathbb{R}^n and then apply the techniques we will have learned to the case of the infinite-dimensional Chern-Simons path integral. The result Z^u will be an expansion for knots, or a "universal finite type invariant".

Talk Video. 

Talk IV: Knotted Trivalent Graphs and Associators. We will find that in order to compute our expansion Z^u on arbitrary knots, it is enough to compute or guess its value on just one specific knotted tetrahedron in \mathbb{R}^2 . This, it turns out, is precisely what is called "a Drinfel'd associator".

The Fourier Transform. $(F: V \rightarrow C) \leftrightarrow (f: V^* \rightarrow C)$ via $F(f) := \int_V f(x)e^{-ix} dx$. Some facts:

- $f(x) = \int_V F(f)e^{ix} dx$.
- $\frac{d}{dx} f = -if$.
- $e^{Q(x)} = e^{-Q(x)}$, where Q is quadratic, $Q(x) = (Lx, x)$ for $L: V \rightarrow V^*$, and $Q^*(y) := (y, L^{-1}y)$. (This is the key point in the proof of the Fourier inversion formula.)

Examples.

Fourier Determinants. If Q and P are matrices and Q is invertible,

$$\int e^{-x^T Q x + x^T P} dx = \int e^{-x^T (Q - P Q^{-1} P) x} dx = \int e^{-x^T (Q - P Q^{-1} P) x} dx = \int e^{-x^T (Q - P Q^{-1} P) x} dx = \int e^{-x^T (Q - P Q^{-1} P) x} dx$$

The Berezin Integral (physics/math language, formulas from Wikipedia: Grassmann Integral). The Berezin integral is linear on functions of anti-commuting variables, and satisfies $\int d\theta = 1$, and $\int \theta d\theta = 0$, so that $\int d\theta^2 = 0$.

Let V be a vector space, $\theta \in V, d\theta \in V^*$ s.t. $(d\theta, \theta) = 1$. Then $f \mapsto \int f d\theta$ is the interior multiplication map $\Lambda V \rightarrow \Lambda V$: $\int f d\theta := \iota_{d\theta} f(\theta)$.

Multiple integration via "Fubini": $\int f(\theta_1) \dots f(\theta_n) d\theta_1 \dots d\theta_n := \int f(\theta_1) \dots \int f(\theta_n) \dots d\theta_n \dots d\theta_1 = \int f(\theta_1) \dots f(\theta_n) \dots d\theta_n \dots d\theta_1$.

Change of variables. If $\theta_i = \theta_i(\xi_j)$, both θ_i and ξ_j are odd, and $J_\xi := \det(\partial \theta_i / \partial \xi_j)$, then

$$\int f(\theta) d\theta = \int f(\theta(\xi)) J_\xi d\xi$$


Given vector spaces V_0 and $W_0, d\theta = \Lambda \theta_0 \in \wedge^{2n} V_0, d\xi = \wedge \xi_0 \in \wedge^{2m} W_0$, and $F: V_0 \rightarrow \wedge^{2m} W_0$. Then F induces a map $F_*: \wedge V_0 \rightarrow \wedge W_0$ and thus

$$\int f d\theta = \int (F_* f) \det \left(\frac{\partial (F \theta_i)}{\partial \xi_j} \right) d\xi$$


Gaussian integration. For an even matrix A and odd vectors θ, ξ , and use "ordinary" perturbation theory.

$$\int e^{-\theta^T A \theta} d\theta = \det(A), \quad \int e^{-\theta^T A \theta + \xi^T \theta} d\theta = \det(A) e^{-\xi^T A^{-1} \xi}$$

add some CFI stuff.

Talk Video. 

Talk V: Back to 4D. We will repeat the 3D story of the previous 3 talks one dimension up, in 4D. Surprisingly, there's more room in 4D, and things get easier, at least when we restrict our attention to "w-knots", or to "simply-knotted 2-knots". But even then there are intricacies, and we try to go beyond simply-knotted, we are completely confused.

Talk Video. 

<http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506/>