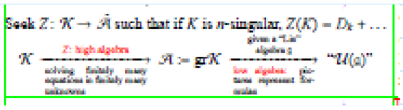
Louvain Day 5 - Back to 4D

June-04-15 7:52 AM

1. What we expect:

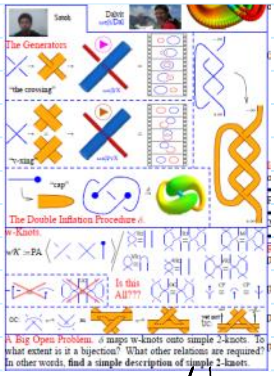
$$\underbrace{\text{Top} \xrightarrow[Z]{gr} \text{comb}}_{\substack{\text{high alg.} \\ \text{or QFT}}} \xrightarrow[\substack{\text{giving} \\ \text{Low algebra}}]{\mathcal{T}_{\mathfrak{g}}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$



Louvain3

2. $wK(\uparrow^n) = PA(\ - - -\ )$

$= CA(\qquad\qquad)$ } make a nice flying tokens picture.



Hamilton

3. Figuring out $gr\,vK$
4. Z using $e^{\wedge}$
5. Bracket rise, trees & wheels.

$$\cdots\quad \mathcal{A}^w(\uparrow^n) \cong \mathcal{U}\left(FL(n)^n_{tb} \ltimes CW(n)\right)$$

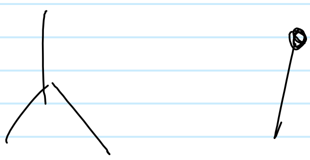
$$\log Z = \} \in FL(n)^n \times CW(n)$$

Has completely explicit formulas using natural FL/CW ops.

6. Low algebra: $\mathfrak{g}$, $I\mathfrak{g}$, $\mathcal{T}_{\mathfrak{g}}$, diff ops.

Too easy so far!

7. wTF :



eq'ns

These are the KV eq'ns!

} follow Goettingen Day 2 as below

8. $a: KTG \longrightarrow wTF$

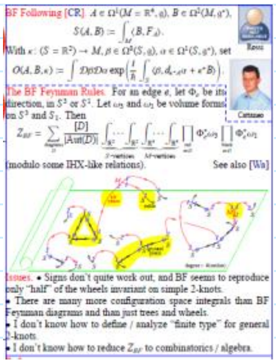
$$\begin{array}{ccc} KTG & \xrightarrow{a} & wTF \\ \downarrow Z^u & & \downarrow Z^w \\ A^u & \xrightarrow{\alpha} & A^w \end{array} \qquad KV \Longleftrightarrow \Phi$$

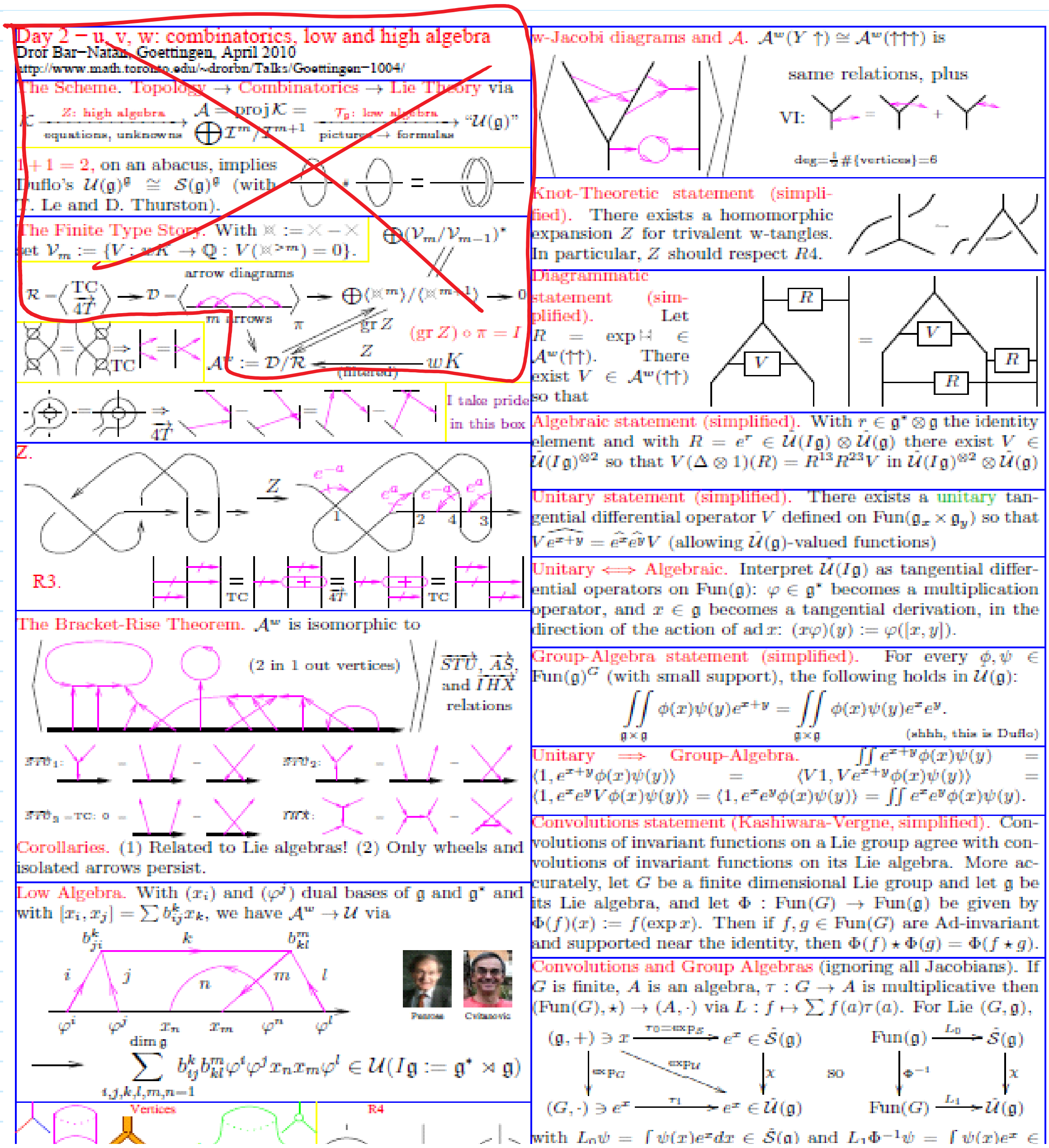
new.

9. BF.

Hamilton

10. RcFs: KBH, WKO1-4.


BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,
$$S(A,B) := \int_M \langle B, F_A \rangle.$$
With $\kappa : (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set
$$O(A,B,\kappa) := \int D\beta D\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^*A}\alpha + \kappa^* B \rangle\right).$$
The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then
$$Z_{BF} = \sum_D \frac{[D]}{|\mathrm{Aut}(D)|} \int \cdots \int \int \cdots \int \prod \Phi_e^* \omega_3 \prod \Phi_e^* \omega_1$$
(modulo some IHX-like relations). See also [Wa]
Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant on simple 2-knots.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define / analyze "finite type" for general 2-knots.
• I don't know how to reduce Z_{BF} to combinatorics / algebra.


Day 2 – u, v, w: combinatorics, low and high algebra
Dror Bar-Natan, Goettingen, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

The Scheme. Topology → Combinatorics → Lie Theory via
$\mathcal{K} \xrightarrow[\text{equations, unknowns}]{Z:\ \text{high algebra}} \mathcal{A} = \mathrm{proj}\,\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1} \xrightarrow[\text{pictures} \to \text{formulas}]{\mathcal{T}_{\mathfrak{g}}:\ \text{low algebra}}$ "$\mathcal{U}(\mathfrak{g})$"

$1+1=2$, on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \cong \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (with T. Le and D. Thurston).

The Finite Type Story. With $\times := \times - \times$ set $\mathcal{V}_m := \{V : w\mathcal{K} \to \mathbb{Q} : V(\times^{>m}) = 0\}$.
$\bigoplus(\mathcal{V}_m/\mathcal{V}_{m-1})^*$
$\mathcal{R} - \langle TC, \overrightarrow{4T} \rangle \to \mathcal{D} - \langle$ arrow diagrams, m arrows $\rangle \to \bigoplus(\times^m)/(\times^{m+1}) \to 0$
$(\mathrm{gr}\, Z) \circ \pi = I$
$\mathcal{A}^v := \mathcal{D}/\mathcal{R} \xleftarrow[\text{(filtered)}]{Z} w\mathcal{K}$
I take pride in this box

Z.

R3.

The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to (2 in 1 out vertices) / $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations

Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Low Algebra. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via
$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is ... same relations, plus VI: ... deg $= \frac{1}{2}\#\{\text{vertices}\} = 6$

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.

Diagrammatic statement (simplified). Let $R = \exp \uparrow\!\!\!\to\!\!\uparrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Unitary $\Longleftrightarrow$ Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$: $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.

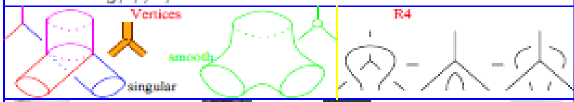
Group-Algebra statement (simplified). For every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$
(shhh, this is Duflo)

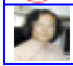
Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y \phi(x)\psi(y)\rangle = \iint e^x e^y \phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \to (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,
$(\mathfrak{g}, +) \ni x \xrightarrow{\tau_0 = \exp_S} e^x \in \hat{\mathcal{S}}(\mathfrak{g})$; $(G, \cdot) \ni e^x \xrightarrow{\tau_1} e^x \in \hat{\mathcal{U}}(\mathfrak{g})$; so $\mathrm{Fun}(\mathfrak{g}) \xrightarrow{L_0} \hat{\mathcal{S}}(\mathfrak{g})$, $\mathrm{Fun}(G) \xrightarrow{L_1} \hat{\mathcal{U}}(\mathfrak{g})$
with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in$

$$\longrightarrow \sum_{i,j,k,l,m,n=1} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$




Kashiwara

Vergne

Alekseev

Torossian

$$\begin{array}{ccc} & \downarrow{\scriptstyle \exp_G} \searrow^{\exp_{\mathcal{U}}} & \downarrow{\scriptstyle \chi} \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \downarrow{\scriptstyle \Phi^{-1}} & & \downarrow{\scriptstyle \chi} \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^x e^y$ $\quad$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$