Louvain2-2

Low and High Algebra in the "u" Case

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

The big picture, "u" case.

Topology — combinatorics

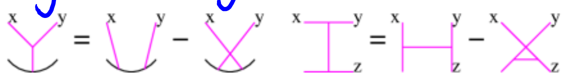
$\mathcal{K} \xrightarrow[\text{expansion}]{Z} \mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}, R} \mathbb{Q}$

$\mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}} U(\mathfrak{g})$

{knots, etc.} — {chord diagrams}/IHX, STU, 4T

high algebra — low algebra

Very low algebra.



[x,y] = xy − yx   [[x,y],z]=[x,[y,z]]−[y,[x,z]]

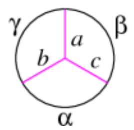
Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [X_a, X_c], X_c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak{g},R} : \quad \text{}$$



$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

Universal Representation Theory.

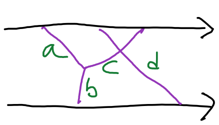
Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $U(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $U(\mathfrak{g})$.
* $\exists\, \Delta : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y \rangle / [x,y] = x$, determine $U(\mathfrak{g})$. Guess a generalization.

Low algebra.

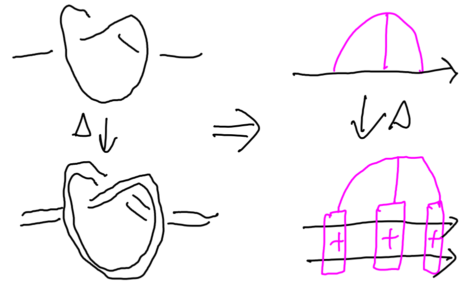
$\mathcal{A}(\uparrow\uparrow) \to U(\mathfrak{g})^{\otimes 2}$ via



$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

& likewise, $\mathcal{A}(\uparrow_n) \to U(\mathfrak{g})^{\otimes n} \Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?



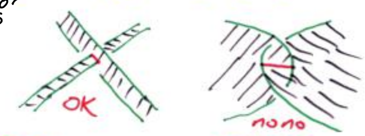
$U(\mathfrak{g}) \xrightarrow{\Delta} U(\mathfrak{g})^{\otimes 2}$

A "Homomorphic Expansion" $Z : \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

$\{\text{Ribbon knots}\} \overset{\text{more or less}}{=} \{u\gamma : \gamma \in \mathcal{K}(\text{O—O}),\ d\gamma = \text{OO}\}$

Where Ribbon means



OK — nono

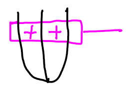
and

Algebraic knot theory:

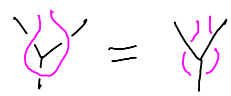
$$\begin{array}{ccc} & \mathcal{K}(\text{OO}) \xrightarrow{Z} \mathcal{A}(\text{OO}) \ni \text{OO} & \\ \mathcal{K}(\text{O—O}) \xrightarrow{d} \mathcal{K}(\text{OO}) & \mathcal{K}(\text{O—O}) \xrightarrow{Z} \mathcal{A}(\text{O—O}) & \\ \mathcal{K}(\text{O—O}) \xrightarrow{u} \mathcal{K}(\text{O}) & \mathcal{K}(\text{O}) \xrightarrow{Z} \mathcal{A}(\text{O}) & \mathcal{A}(\text{O—O}) \xrightarrow{u} \mathcal{A}(\text{O}) \end{array}$$

So $Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(\text{OO})\} \subset \mathcal{A}(\text{O—O})$

VI



$= 0$, follows from



Talk II: From Knots to Lie Algebras. Why on Earth should knots be related to Lie algebras? The former are squishy and irregular, the latter are symmetric and rigid. They should know nothing of each other. Yet as we shall see, the natural target space for expansions for knots is in some sense, "the universal dual" of all (metrized) Lie algebras.