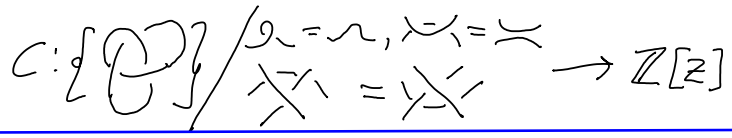
Louvain2-1

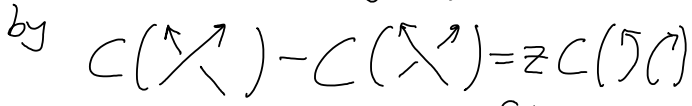
The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

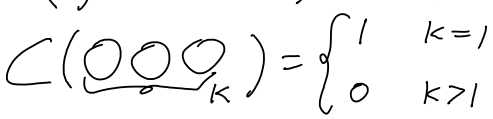
Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.



Example. The Conway polynomial is given by

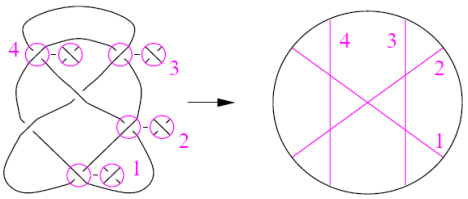


and



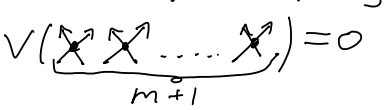
Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

Definition. Any V: {knots} → Abelian Group A can be extended to "knots w/ double points"



using $V(\times) = V(\times) - V(\times)$. (Think "differentiation")

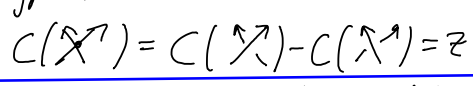
Definition. V is of type m if always



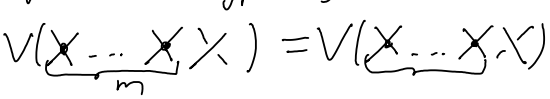
(think "polynomial")

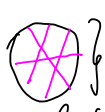
Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

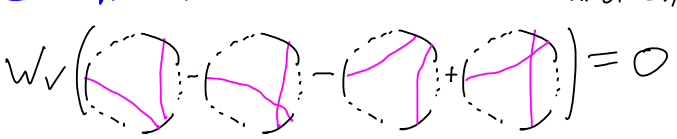
Proof.  □

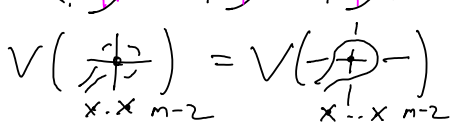
Let V be of type m; then $V^{(m)}$ is constant:



So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: W_V: {  } → A

Claim. W_V satisfies the 4T relation:



Proof. 

□

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.

2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

Theorem. (The Fundamental Theorem)

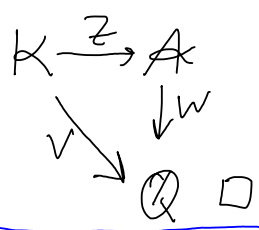
M. Kontsevich

Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W = W_V$

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}_m^r$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

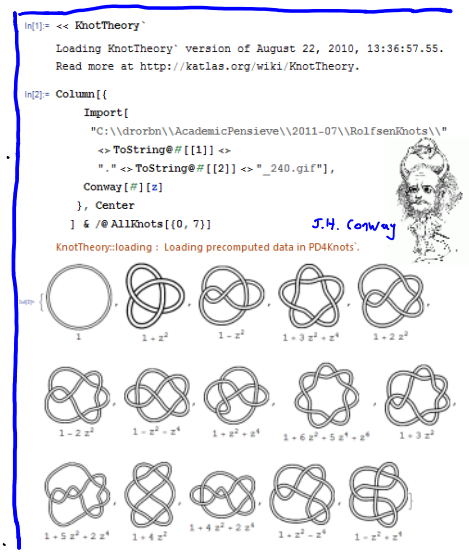
Proposition. The fundamental thm holds iff there exists an expansion: $Z: \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.

Proof.

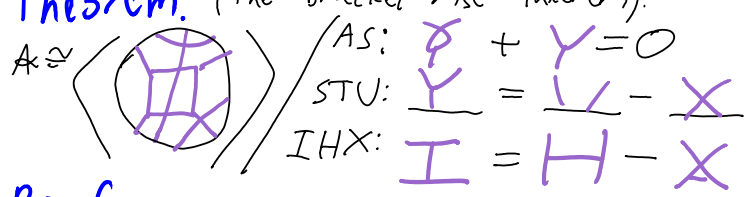


□

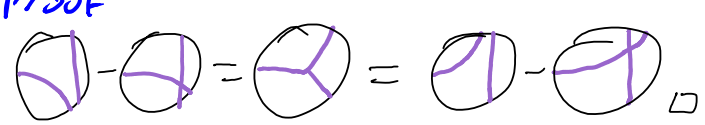
Note. Z is precisely an "Expansion" in the sense of yesterday.



Theorem. (The "bracket-rise" theorem).



Proof



□

Also see my old paper, "On the Vassiliev Knot Invariants" (a web search will find…)