Affine group maps

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(150624) "Set function $\varphi\colon G \to H$ is affine" means $\varphi(I^n_G) \subset I^n_H$. Morphisms are affine, "sorting" on FG is affine. A category?

Is $Id_1\colon G \ltimes H \to G\times H$ affine?

Is $Id_2\colon G\times H \to G\ltimes H$ affine?

Claim If $\varphi_i\colon G_i \to H$ are affine, then the map $G_1\times G_2 \to H$ by
$(g_1, g_2) \mapsto \varphi_1(g_1)\varphi_2(g_2)$ is affine.

This proves both that "sorting" is affine, and that Id_2 is affine.

Prop If $\underbrace{G \to G^{ab}}_{\text{f.g.}}$ has the property that every cyclic subgroup of G^{ab} is the isomorphic image of a cyclic subgroup of G, then $G\to G^{ab}$ has an affine section.

Pf Follows from the above claim and from the structure theorem for f.g. Abelian groups.

Claim If $G \ltimes H$ is almost-direct, then $Id_1 : G \ltimes H \to G \times H$ is affine.

~~Prop~~ (2) $\varphi : \mathbb{Z} \to FG$ is affine iff

$\varphi(n) = hg^n$ for some $(g,h) \in FG$.

Pf WLOG, $h=1$; i.e., $\varphi(0)=1$.

False! $\varphi(n) = x^n y^n$ is affine.

$1 - 2xy + x^2y^2 =$ $\quad$ $1-xy = (1-x) + (x-xy)$

$= \varphi((1-x+x(1-y))^2) = \varphi((1-x)^2) + 2\varphi((1-x)x(1-y))$
$+ \varphi(x^2(1-y)^2)$

$= (1-x)^2 + 2(x-x^2)(1-y) + x^2(1-y^2)$

Indeed,

$= 1 - \cancel{2x} + \cancel{x^2} + \cancel{2x} - \cancel{2x^2} - 2xy + 2x^2y^2$
$+ \cancel{x^2} - x^2y^2$ ✓

Similarly,

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In[1]:= (1 - x)^3 + 3 x (1 - x)^2 (1 - y) + 3 x^2 (1 - x) (1 - y)^2 + x^3 (1 - y)^3 // Expand
Out[1]= 1 - 3 x y + 3 x^2 y^2 - x^3 y^3
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Q: Will $\varphi_{1,2} : \mathbb{Z} \to FG$ by

$\varphi_1(n) = (x,y)^{n^2}$

$\varphi_2(n) = ((x,y)^n (x,z)^n)^n$

be affine?