Pensieve header: One-Co computations; continues pensieve://2015-05/.

The bracket

In the basis {a[f,i,j], γ[f,i,j], γ[f,i,j,k], γa[f,i,j,k,l]}.

Generalities

```
CF[expr_] := expr /. {
   a[f_, i_, j_] :> Interpretation[StringForm["a[``,````]", f, i, j], a[f, i, j]],
   γ[f_, i_, j_] :> Interpretation[StringForm["γ[``,````]", f, i, j], γ[f, i, j]],
   γ[f_, i_, j_, k_] :> Interpretation[StringForm["γ[``,``````]", f, i, j, k], γ[f, i, j, k]],
   γa[f_, i_, j_, k_, l_] :>
    Interpretation[StringForm["γa[``,````````]", f, i, j, k, l], γa[f, i, j, k, l]]
  }
```

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
```

```
Simp[expr_] := Simplify[expr];
S[γ[f_, j_, k_, k_]] = 0;
S[γ[f_, j_, k_, l_]] /; OrderedQ[{l, k}] ∧ DQ[k, l] := γ[-f, j, l, k] // S;
S[γa[f_, i_, j_, i_, k_]] /; OrderedQ[{k, j}] ∧ DQ[k, j] := γ[b_i f, i, j, k] + γa[f, i, k, i, j] // S;
(*S[γa[f_,i_,j_,k_,l_]] /;
   (DQ[i,j,k,l]∧(OrderedQ[{i,j,k,l}]∨OrderedQ[{i,k,j,l}]∨OrderedQ[{i,l,j,k}])) :=
  γ[b_k f,i,j,l]-γ[b_i f,k,j,l]-γa[f,k,l,i,j]+γa[f,i,l,k,j]+γa[f,k,j,i,l];*)
S[β[f_]] := β[Simp[f]];
S[a[i_, j_]] := a[i, j];
S[a[f_, i_, j_]] := a[Simp[f], i, j];
S[γ[f_, is__]] := γ[Simp[f], is];
S[γa[f_, is__]] := γa[Simp[f], is];
S[expr_] := expr /. (λ_β | λ_a | λ_γ | λ_γa) :> S[λ];
```

$$L_1 \gamma_{234} = \gamma_{23} a_{14} - \gamma_{24} a_{13}$$

```
S[γa[f_, i_, j_, k_, l_]] /; DQ[i, j, k, l] ∧ OrderedQ[{l, j}] := -γ[b_k f, i, l, j] + γa[f, i, l, k, j];
```

```
β[0] := 0;
β /: β[f_] + β[g_] := β[f + g];
β /: -β[f_] := β[-f] // S;
a[0, __] := 0;
a /: a[f_, i_, j_] + a[g_, i_, j_] := a[f + g, i, j];
a /: -a[f_, is__] := a[-f, is] // S;
γ[0, __] := 0;
γ /: γ[f_, is__] + γ[g_, is__] := γ[f + g, is];
γ /: -γ[f_, is__] := γ[-f, is] // S;
γa[0, __] := 0;
γa /: γa[f_, is__] + γa[g_, is__] := γa[f + g, is];
γa /: -γa[f_, is__] := γa[-f, is] // S;
```

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

Specific Brackets

Generalities:

```
B[a[j_, k_], a[i_, j_]] /; DQ[i, j, k] := -B[a[i, j], a[j, k]];
B[a[f_, j_, k_], a[g_, l_, m_]] := Plus[
    B[a[j, k], a[l, m]] /. {a[h_, is__] :> a[f g h, is], γ[h_, is__] :> γ[f g h, is]},
    γa[f (∂_{b_j} g - ∂_{b_k} g), j, k, l, m] + γa[g (∂_{b_m} f - ∂_{b_l} f), l, m, j, k]
   ] // S;
```

Vanishing brackets:

```
B[a[j_, k_], a[l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, j_, l_]] /; DQ[j, k, l] := 0;
B[a[f_, j_, k_], γ[g_, j_, l_, m_]] /; DQ[j k l m] := 0;
B[a[f_, i_, j_], γ[g_, k_, l_, m_]] /; DQ[i, j, k, l, m] := 0;
```

Non-vanishing brackets:

```
B[a[f_ j_ k_], β[g_]] := γ[f (∂_{b_j} g - ∂_{b_k} g), j, k] // S;
B[a[j_ k_], a[j_ l_]] /; DQ[j, k, l] := γ[1, j, k, l] // S;
B[a[j_ k_], a[i_ k_]] /; DQ[i j k] := S[a[b_i, j, k] - a[b_j, i, k]];
B[a[j_ k_], a[k_ l_]] /; DQ[j k l] := S[a[b_j, k, l] - a[b_k, j, l] - γ[1, j, k, l]];
B[a[f_ j_ k_], γ[g_ i_ k_]] /; DQ[i j k] := S@ γ[-b_j f g, i, k];
B[a[f_ j_ k_], γ[g_ i_ j_]] /; DQ[i, j, k] := S@γ[b_j f g, i, k];
B[a[f_ j_ k_], γ[g_ k_ l_]] /; DQ[j, k, l] := S[γ[b_j f g, k, l] - γ[b_k f g, j, l]];
B[a[f_ j_ k_], γ[g_ j_ k_]] := γ[-b_j f g, j, k] // S;
B[a[f_ j_ k_], γ[g_ i_ j_ l_]] /; DQ[i j k l] := γ[b_j f g, i, k, l] + γa[f g, i, l, j, k] // S;
B[a[f_ l_ k_], γ[g_ i_ j_ l_]] /; DQ[i, j, k, l] := γ[-b_l f g, i, k, j] + γa[-f g, i, j, l, k] // S;
B[a[f_ j_ k_], γ[g_ k_ l_ m_]] /; DQ[j, k, l, m] := γ[-b_k f g, j, l, m] + γ[b_j f g, k, l, m] // S;
B[a[f_ j_ k_], γ[g_ n_ i_ k_]] /; DQ[n, i, j, k] := γ[-b_j f g, n, i, k] + γa[f g, n, i, j, k] // S;
B[a[f_ j_ i_], γ[g_ n_ i_ k_]] /; DQ[n, i, j, k] := γ[b_j f g, n, k, i] + γa[-f g, n, k, j, i] // S;
B[a[f_ j_ k_], γ[g_ j_ k_ l_]] /; DQ[j k l] := S@γa[-f g, j, k, j, l];
B[a[f_ j_ l_], γ[g_ j_ k_ l_]] /; DQ[j k l] := S@γa[f g, j, l, j, k];
B[a[f_ j_ k_], γ[g_ i_ j_ k_]] /; DQ[i, j, k] :=
  S[γ[-b_j f g, i, j, k] + γa[f g, i, j, j, k] + γa[f g, i, k, j, k]];
B[a[f_ k_ j_], γ[g_ i_ j_ k_]] /; DQ[i, j, k] := S[γ[b_k f g, i, k, j] + γa[-f g, i, k, k, j] + γa[-f g, i, j, k, j]];
```

[a, γa] brackets:

```
B[x_a, γa[f_, i_, j_, m_, n_]] := Plus[
   B[x, γ[f, i, j]] /. γ[g_, k_, l_] :> γa[g, k, l, m, n],
   B[x, a[1, m, n]] /. {a[g_, k_, l_] :> γa[f g, i, j, k, l], _γ | _γa :> 0}
  ] // S
```

[β,a], [γ, a], [γ,γ] brackets:

```
B[x_β | x_γ | x_γa, y_a] := -B[y, x];
B[_β | _γ | _γa, _β | _γ | _γa] := 0;
```

```
tests = {β[f[b_j, b_k]], γ[1, i, j], γ[1, i, k], γ[1, i, l], γ[1, j, k], γ[1, j, l], γ[1, k, l],
   γ[1, j, l, m], γ[1, i, j, l], γ[1, i, k, l], γ[1, k, l, m], γ[1, j, k, l], γ[1, i, j, k],
   a[1, j, k], a[1, j, l], a[1, i, k], a[1, i, j], a[1, k, l], a[1, l, m], γa[1, j, k, j, l]};
shows = Complement[tests, {γ[1, i, l], γ[1, j, l], γ[1, j, l, m], a[1, j, k], a[1, l, m]}];
```

```
ColumnForm[(# → B[a[1, j, k], #]) & /@ shows] // CF
```

OneCoBrackets

```
a[1,ij] → a[-b_i,jk] + a[b_j,ik] + γ[1,ijk]
a[1,ik] → a[b_i,jk] + a[-b_j,ik]
a[1,jl] → γ[1,jkl]
a[1,kl] → a[b_j,kl] + a[-b_k,jl] + γ[-1,jkl]
β[f[b_j, b_k]] → γ[-f^(0,1)[b_j, b_k] + f^(1,0)[b_j, b_k],jk]
γ[1,ij] → γ[b_j,ik]
γ[1,ik] → γ[-b_j,ik]
γ[1,jk] → γ[-b_j,jk]
γ[1,kl] → γ[b_j,kl] + γ[-b_k,jl]
γ[1,ijk] → γ[-b_j,ijk] + γa[1,ijjk] + γa[1,ikjk]
γ[1,ijl] → γa[1,ikjl]
γ[1,ikl] → γa[-1,ikjl]
γ[1,jkl] → γa[-1,jkjl]
γ[1,klm] → γ[b_j,klm] + γ[-b_k,jlm]
γa[1,jkjl] → γa[-b_j,jkjl]
```

Testing Jacobi and Anti-Symmetry

```
t1 = B[a[1, 1, 2], γ[b2, 2, 3, 4]]
```

$\gamma[b_1 b_2, 2, 3, 4] + \gamma[-b_2^2, 1, 3, 4]$

```
t2 = B[a[1, 1, 2], γa[1, 2, 3, 2, 4]]
```

$\gamma a[2 b_1, 2, 3, 2, 4] + \gamma a[-b_2, 1, 3, 2, 4] + \gamma a[-b_2, 2, 3, 1, 4]$

```
t3 = B[a[1, 1, 2], γa[1, 2, 4, 2, 3]]
```

$\gamma[-b_1 b_2, 2, 3, 4] + \gamma[b_2^2, 1, 3, 4] + \gamma a[2 b_1, 2, 3, 2, 4] + \gamma a[-b_2, 1, 3, 2, 4] + \gamma a[-b_2, 2, 3, 1, 4]$

```
t1 - t2 + t3
```

0

```
FormalPlusBasis[n_, f_] := Module[{ff},
  ff = f @@ Table[b_i, {i, n}];
  Flatten@{
    Table[a[ff, i, j], {i, n - 1}, {j, i + 1, n}],
    Table[γ[ff, i, j], {i, n - 1}, {j, i + 1, n}],
    Table[γ[ff, i, j, k], {i, n - 2}, {j, i + 1, n - 1}, {k, j + 1, n}],
    Table[γa[ff, i, j, k, l], {i, n - 1}, {j, i + 1, n}, {k, n - 1}, {l, k + 1, n}]
   } /. 1[___] → 1
 ];
```

```
FormalPlusBasis[3, f]
```

$\{a[f[b_1, b_2, b_3], 1, 2], a[f[b_1, b_2, b_3], 1, 3], a[f[b_1, b_2, b_3], 2, 3],$
$\gamma[f[b_1, b_2, b_3], 1, 2], \gamma[f[b_1, b_2, b_3], 1, 3], \gamma[f[b_1, b_2, b_3], 2, 3], \gamma[f[b_1, b_2, b_3], 1, 2, 3],$
$\gamma a[f[b_1, b_2, b_3], 1, 2, 1, 2], \gamma a[f[b_1, b_2, b_3], 1, 2, 1, 3], \gamma a[f[b_1, b_2, b_3], 1, 2, 2, 3],$
$\gamma a[f[b_1, b_2, b_3], 1, 3, 1, 2], \gamma a[f[b_1, b_2, b_3], 1, 3, 1, 3], \gamma a[f[b_1, b_2, b_3], 1, 3, 2, 3],$
$\gamma a[f[b_1, b_2, b_3], 2, 3, 1, 2], \gamma a[f[b_1, b_2, b_3], 2, 3, 1, 3], \gamma a[f[b_1, b_2, b_3], 2, 3, 2, 3]\}$

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // S;
   If[as === 0, as, {x1, x2} → as]
  ];
Outer[
 AS,
 FormalPlusBasis[3, f],
 FormalPlusBasis[3, g]
]
```

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

```
Jacobi[x1_, x2_, x3_] := Module[{Jac},
   Jac = S[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
   If[Jac === 0, Jac, {x1, x2, x3} → Jac]
  ];
```

```
JacErrors = DeleteCases[
  bas1 = FormalPlusBasis[4, f];
  bas2 = FormalPlusBasis[4, g];
  bas3 = FormalPlusBasis[4, h];
  Flatten[
   Table[Jacobi[bas1⟦i⟧, bas2⟦j⟧, bas3⟦k⟧],
    {i, Length[bas1] - 1}, {j, i + 1, Length@bas2}, {k, i + 1, Length@bas3}]
  ],
  0]
```

$RecursionLimit::reclim : Recursion depth of 1024 exceeded. ≫

$RecursionLimit::reclim : Recursion depth of 1024 exceeded. ≫

$RecursionLimit::reclim : Recursion depth of 1024 exceeded. ≫

General::stop : Further output of $RecursionLimit::reclim will be suppressed during this calculation. ≫

```
{{a[f[b1, b2, b3, b4], 1, 2], a[g[b1, b2, b3, b4], 2, 3], a[h[b1, b2, b3, b4], 3, 4]} → Jac$747,
 {a[f[b1, b2, b3, b4], 1, 2], a[g[b1, b2, b3, b4], 3, 4], a[h[b1, b2, b3, b4], 2, 3]} → Jac$1458,
 {a[f[b1, b2, b3, b4], 1, 3], a[g[b1, b2, b3, b4], 2, 3], γa[h[b1, b2, b3, b4], 3, 4, 1, 2]} → Jac$3947,
 {a[f[b1, b2, b3, b4], 1, 3], γa[g[b1, b2, b3, b4], 3, 4, 1, 2], a[h[b1, b2, b3, b4], 2, 3]} → Jac$6054,
 {a[f[b1, b2, b3, b4], 1, 4], a[g[b1, b2, b3, b4], 2, 3], γa[h[b1, b2, b3, b4], 3, 4, 1, 2]} → Jac$6396,
 {a[f[b1, b2, b3, b4], 1, 4], γa[g[b1, b2, b3, b4], 3, 4, 1, 2], a[h[b1, b2, b3, b4], 2, 3]} → Jac$8460,
 {a[f[b1, b2, b3, b4], 2, 3], a[g[b1, b2, b3, b4], 2, 4], γa[h[b1, b2, b3, b4], 3, 4, 1, 2]} → Jac$8796,
 {a[f[b1, b2, b3, b4], 2, 3], a[g[b1, b2, b3, b4], 3, 4], γa[h[b1, b2, b3, b4], 3, 4, 1, 2]} → Jac$8844,
 {a[f[b1, b2, b3, b4], 2, 3], γa[g[b1, b2, b3, b4], 3, 4, 1, 2], a[h[b1, b2, b3, b4], 2, 4]} → Jac$10770,
 {a[f[b1, b2, b3, b4], 2, 3], γa[g[b1, b2, b3, b4], 3, 4, 1, 2], a[h[b1, b2, b3, b4], 3, 4]} → Jac$10771}
```

```
DeleteCases[
 Flatten[Outer[
   Jacobi,
   FormalPlusBasis[4, f],
   FormalPlusBasis[4, g],
   FormalPlusBasis[4, h]
  ]],
 0]
```

Testing representations

The $R\langle\beta\gamma_2\rangle$ representation

```
Outer[
   Function[{x, y, r}, {x, y, r} → S[B[B[x, y], r] - B[x, B[y, r]] + B[y, B[x, r]]]],
   {a[1, 1, 2], a[1, 1, 3], a[1, 2, 3]},
   {a[1, 1, 2], a[1, 1, 3], a[1, 2, 3]},
   {β[f[b₁, b₂, b₃]]}
  ] // Flatten // ColumnForm
```

```
{a[1, 1, 2], a[1, 1, 2], β[f[b1, b2, b3]]} → 0
{a[1, 1, 2], a[1, 1, 3], β[f[b1, b2, b3]]} → 0
{a[1, 1, 2], a[1, 2, 3], β[f[b1, b2, b3]]} → 0
{a[1, 1, 3], a[1, 1, 2], β[f[b1, b2, b3]]} → 0
{a[1, 1, 3], a[1, 1, 3], β[f[b1, b2, b3]]} → 0
{a[1, 1, 3], a[1, 2, 3], β[f[b1, b2, b3]]} → 0
{a[1, 2, 3], a[1, 1, 2], β[f[b1, b2, b3]]} → 0
{a[1, 2, 3], a[1, 1, 3], β[f[b1, b2, b3]]} → 0
{a[1, 2, 3], a[1, 2, 3], β[f[b1, b2, b3]]} → 0
```

The Adjoint action

```
AutoAd[x_][y_] := Module[{pows, states, s, seq, sh = 5, sf, t1, n},
  pows = NestList[B[x, #] &, y, 15];
  states = Union[Cases[pows, s_a | s_β | s_γ | s_γa :> ReplacePart[s, 1 → _], ∞]];
  Sum[
   seq = Cases[{#}, s, ∞] & /@ pows;
   seq = Replace[seq, {{_[f_, ___]} :> f, {} → 0}, {1}];
   sf = FindSequenceFunction[Drop[seq, sh]];
   ReplacePart[s, 1 → FullSimplify[Sum[seq[[n + 1]]/n!, {n, 0, sh-1}] + Sum[sf[n + 1 - sh]/n!, {n, sh, ∞}]]],
   {s, states}]
 ];
AutoAd1[x_][y_] := Module[{pows, states, s, seq, sh, sf, t1, n},
 pows = NestList[B[x, #] &, y, 15];
 states = Union[Cases[pows, s_a | s_β | s_γ | s_γa :> ReplacePart[s, 1 → _], ∞]];
 Sum[
  seq = Cases[{#}, s, ∞] & /@ pows;
  sh = 0; While[seq[[1]] === {}, ++sh; seq = Rest[seq]];
  seq = Replace[seq, {{_[f_, ___]} :> f, {} → 0}, {1}];
  sf = Replace[seq, {
     {t1_, 0 ...} :> (t1 * KroneckerDelta[1, #] &),
     seq_ :> FindSequenceFunction[seq]
    }];
  ReplacePart[s, 1 → FullSimplify[Sum[sf[n + 1 - sh]/n!, {n, sh, ∞}]]],
  {s, states}
 ]
]
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)
```

```
Ad[a[t_, j_, k_]][β[f_]] /; FreeQ[t, b_] := β[f] + γ[((ⅇ^(-t b_j) - 1) (∂_{b_k} f - ∂_{b_j} f))/b_j, j, k];
Ad[a[t_, j_, k_]][γ[f_, j_, k_]] /; FreeQ[t, b_] := γ[ⅇ^(-t b_j) f, j, k];
Ad[a[t_, j_, k_]][γ[f_, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] := γ[f, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := γ[ⅇ^(-t b_j) f, i, k];
Ad[a[t_, j_, k_]][γ[f_, i_, j_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := γ[f, i, j] + γ[f - ⅇ^(-t b_j) f, i, k];
Ad[a[t_, j_, k_]][γ[f_, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] := γ[ⅇ^(t b_j) f, k, l] + γ[((1 - ⅇ^(t b_j)) f b_k)/b_j, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] := γ[f, i, l];
Ad[a[t_, j_, k_]][γ[f_, j_, l_, m_]] /; DQ[j, l, k, m] ∧ FreeQ[t, b_] := γ[f, j, l, m];
Ad[a[t_, j_, k_]][γ[f_, i_, j_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] :=
  γ[f, i, j, l] + γ[f - ⅇ^(-t b_j) f, i, k, l] + γa[(f - ⅇ^(-t b_j) f)/b_j, i, l, j, k];
Ad[a[t_, j_, k_]][γ[f_, i_, k_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(-t b_j) f, i, k, l] + γa[((ⅇ^(-t b_j) - 1) f)/b_j, i, l, j, k];
Ad[a[t_, j_, k_]][γ[f_, k_, l_, m_]] /; DQ[j, k, l, m] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(t b_j) f, k, l, m] + γ[((1 - ⅇ^(t b_j)) f b_k)/b_j, j, l, m];
Ad[a[t_, j_, k_]][γ[f_, j_, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  γ[f, j, k, l] + γa[((-1 + ⅇ^(-t b_j)) f)/b_j, j, k, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, j_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(-t b_j) f, i, j, k] + γa[(1 - ⅇ^(-t b_j))/b_j f, i, j, j, k] + γa[(1 - ⅇ^(-t b_j))/b_j f, i, k, j, k];
Ad[a[t_, j_, k_]][a[1, j_, k_]] /; FreeQ[t, b_] := a[1, j, k];

Ad[a[t_, j_, k_]][a[1, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
 a[1, j, l] + γ[t, j, k, l] + γa[(1 - ⅇ^(-t b_j) - t b_j)/b_j^2, j, k, j, l];

Ad[a[t_, j_, k_]][a[1, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := a[ⅇ^(-t b_j), i, k] + a[((1 - ⅇ^(-t b_j)) b_i)/b_j, j, k] +
  γa[(ⅇ^(-2 t b_j) b_i (2 ⅇ^(t b_j) t b_j + 1 - ⅇ^(2 t b_j)))/b_j^3, j, k, j, k] + γa[(ⅇ^(-2 t b_j) (ⅇ^(t b_j) (1 - t b_j) - 1))/b_j^2, j, k, i, k];
Ad[a[t_, j_, k_]][a[1, i_, j_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := a[1, i, j] + a[1 - ⅇ^(-t b_j), i, k] +
  a[((ⅇ^(-t b_j) - 1) b_i)/b_j, j, k] + γ[(1 - ⅇ^(-t b_j))/b_j, i, j, k] + γa[(b_i (1 - 2 ⅇ^(-t b_j) t b_j - ⅇ^(-2 t b_j)))/b_j^3, j, k, j, k] +
  γa[(ⅇ^(-t b_j) + t b_j - 1)/b_j^2, i, j, j, k] + γa[(ⅇ^(-t b_j) + t b_j - 1)/b_j^2, i, k, j, k] + γa[(ⅇ^(-2 t b_j) (1 + ⅇ^(t b_j) (-1 + t b_j)))/b_j^2, j, k, i, k];
Ad[a[t_, j_, k_]][a[1, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[ⅇ^(t b_j), k, l] + a[((1 - ⅇ^(t b_j)) b_k)/b_j, j, l] + γ[(t b_j b_k + (1 - ⅇ^(t b_j)) (b_j + b_k))/b_j^2, j, k, l] +
  γa[(1 + ⅇ^(t b_j) (t b_j - 1))/b_j^2, j, k, k, l] + γa[(ⅇ^(-t b_j) b_j + ⅇ^(t b_j) (b_j + 2 b_k - t b_j b_k) - 2 b_j - 2 b_k - t b_j b_k)/b_j^3, j, k, j, l];
Ad[a[t_, j_, k_]][a[1, l_, m_]] /; DQ[j, k, l, m] ∧ FreeQ[t, b_] := a[1, l, m];
(*Ad[β[f_]][a[1, i_,j_]] := a[1,i,j]+γ[∂_{b_j}f-∂_{b_i}f,i,j];*)
Ad[x_β | x_γ | x_γa][y_] := y + B[x, y];
Ad[β[f_]][t_β | t_γ | t_γa] := t;
```

```
AdTests[a[t, j, k]] =
  {β[f[bj, bk]], γ[f, i, j], γ[f, i, k], γ[f, j, l], γ[f, k, l], γ[f, i, j, l], γ[f, j, k, l],
   γ[f, i, j, k], γ[f, i, k, l], γ[f, k, l, m], a[1, j, k], a[1, j, l], a[1, i, k], a[1, i, j], a[1, k, l]};

S[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ AdTests[a[t, j, k]]

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

AutoAd[β[f[bi, bj]]][a[g, i, j]] - Ad[β[f[bi, bj]]][a[g, i, j]] // S

0

Ad[x_][y_Plus] := Ad[x] /@ y;
(Ad[x_][a[f_ , i_, j_]] /; f =!= 1) := Module[{Adf, g, γs},
   Adf = β[f] // Ad[x];
   Adf /. β[h_] :> (g = h);
   γs = Cases[{Adf - β[g]}, _γ, ∞];
   S[(a[1, i, j] // Ad[x]) /. {
      a[h_, k_, l_] :> Plus[
        a[g * h, k, l],
        Total[γs /. γ[hh_, m_, n_] :> γa[h * hh, m, n, k, l]]
       ],
      γ[h_, is__] :> γ[g * h, is],
      γa[h_, is__] :> γa[g * h, is]
     }]
  ];
Ad[x_][γa[f_, i_, j_, k_, l_]] := S[
   Expand[(γ[f, i, j] // Ad[x]) (a[1, k, l] // Ad[x])] /. {
     γ[g_, m_, n_] a[h_, p_, q_] :> γa[g * h, m, n, p, q],
     _γ * (_γ | _γa) → 0
    }
  ];

Ad[x_][y_] := (Print["Ad not yet defined on ", {x, y}]; y);
```

OneCoAd

```
Print[CF[# -> Ad[a[t, j, k]][#]]] & /@ shows;
```

$$a[1,ij] \to a[1,ij] + a[1-e^{-t b_j},ik] + a[\frac{(-1+e^{-t b_j}) b_i}{b_j},jk] + \gamma[\frac{1-e^{-t b_j}}{b_j},ijk] + \gamma a[\frac{-1+e^{-t b_j}+t b_j}{b_j^2},ijjk] + \gamma a[\frac{-1+e^{-t b_j}+t b_j}{b_j^2},ikjk] + \gamma a[\frac{b_i (1-e^{-2 t b_j}-2 e^{-t b_j} t b_j)}{b_j^3},jkjk] + \gamma a[\frac{e^{-2 t b_j} (1+e^{t b_j} (-1+t b_j))}{b_j^2},jkik]$$

OneCoAd

$$a[1,ik] \to a[e^{-t b_j},ik] + a[\frac{(1-e^{-t b_j}) b_i}{b_j},jk] + \gamma a[\frac{e^{-2 t b_j} b_i (1-e^{2 t b_j}+2 e^{t b_j} t b_j)}{b_j^3},jkjk] + \gamma a[\frac{e^{-2 t b_j} (-1+e^{t b_j} (1-t b_j))}{b_j^2},jkik]$$

OneCoAd

$$a[1,jl] \to a[1,jl] + \gamma[t,jkl] + \gamma a[\frac{1-e^{-t b_j}-t b_j}{b_j^2},jkjl]$$

OneCoAd

$$a[1,kl] \to a[e^{t b_j},kl] + a[\frac{(1-e^{t b_j}) b_k}{b_j},jl] + \gamma[\frac{t b_j b_k + (1-e^{t b_j}) (b_j+b_k)}{b_j^2},jkl] + \gamma a[\frac{1+e^{t b_j} (-1+t b_j)}{b_j^2},jkkl] + \gamma a[\frac{-2 b_j + e^{-t b_j} b_j - 2 b_k - t b_j b_k + e^{t b_j} (b_j + 2 b_k - t b_j b_k)}{b_j^3},jkjl]$$

OneCoAd

$$\beta[f[b_j, b_k]] \to \gamma[\frac{(-1+e^{-t b_j}) (f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k])}{b_j},jk] + \beta[f[b_j, b_k]]$$

OneCoAd

$$\gamma[1,ij] \to \gamma[1,ij] + \gamma[1-e^{-t b_j},ik]$$

OneCoAd

$$\gamma[1,ik] \to \gamma[e^{-t b_j},ik]$$

OneCoAd

$$\gamma[1,jk] \to \gamma[e^{-t b_j},jk]$$

OneCoAd

$$\gamma[1,kl] \to \gamma[e^{t b_j},kl] + \gamma[\frac{(1-e^{t b_j}) b_k}{b_j},jl]$$

OneCoAd

$$\gamma[1,ijk] \to \gamma[e^{-t b_j},ijk] + \gamma a[\frac{1-e^{-t b_j}}{b_j},ijjk] + \gamma a[\frac{1-e^{-t b_j}}{b_j},ikjk]$$

OneCoAd

$$\gamma[1,ijl] \to \gamma[1,ijl] + \gamma[1-e^{-t b_j},ikl] + \gamma a[\frac{1-e^{-t b_j}}{b_j},iljk]$$

OneCoAd

$$\gamma[1,ikl] \to \gamma[e^{-t b_j},ikl] + \gamma a[\frac{-1+e^{-t b_j}}{b_j},iljk]$$

OneCoAd

$$\gamma[1,jkl] \to \gamma[1,jkl] + \gamma a[\frac{-1+e^{-t b_j}}{b_j},jkjl]$$

OneCoAd

$$\gamma[1,klm] \to \gamma[e^{t b_j},klm] + \gamma[\frac{(1-e^{t b_j}) b_k}{b_j},jlm]$$

OneCoAd

$$\gamma a[1,jkjl] \to \gamma a[e^{-t b_j},jkjl]$$

The semi group properties

```
Module[{t1, t2},
  t1 = Ad[a[t, j, k]][#] /. (h : (a | β | γ | γa))[c_, r___] :> h[SeriesCoefficient[c, {t, 0, 1}], r];
  t2 = B[a[1, j, k], #];
  # → S[t1 == t2]] & /@ tests // ColumnForm
```

```
β[f[b_j, b_k]] → True
γ[1, i, j] → True
γ[1, i, k] → True
γ[1, i, l] → True
γ[1, j, k] → True
γ[1, j, l] → True
γ[1, k, l] → True
γ[1, j, l, m] → True
γ[1, i, j, l] → True
γ[1, i, k, l] → True
γ[1, k, l, m] → True
γ[1, j, k, l] → True
γ[1, i, j, k] → True
a[1, j, k] → True
a[1, j, l] → True
a[1, i, k] → True
a[1, i, j] → True
a[1, k, l] → True
a[1, l, m] → True
γa[1, j, k, j, l] → True
```

```
Module[{t1, t2},
  t1 =
   Ad[β[t f[b_i, b_j, b_k]]][#] /. (h : (a | β | γ | γa))[c_, r___] :> h[SeriesCoefficient[c, {t, 0, 1}], r];
  t2 = B[β[f[b_i, b_j, b_k]], #];
  # → S[t1 == t2]] & /@ {a[g[b_i, b_j, b_k], i, j]} // ColumnForm
```

```
a[g[b_i, b_j, b_k], i, j] → True
```

```
Module[{t1, t2},
  t1 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
  t2 = # // Ad[a[t + s, j, k]];
  # → t1 - t2 // S] & /@ tests // ColumnForm
```

```
β[f[b_j, b_k]] → 0
γ[1, i, j] → 0
γ[1, i, k] → 0
γ[1, i, l] → 0
γ[1, j, k] → 0
γ[1, j, l] → 0
γ[1, k, l] → 0
γ[1, j, l, m] → 0
```

$\gamma[1, i, j, l] \to \gamma a\left[\frac{1-e^{-s b_j}}{b_j} + \frac{e^{-t b_j}(-1+e^{-s b_j})}{b_j} + \frac{(-1+e^{-s b_j})(1-e^{-t b_j})}{b_j}, i, k, j, l\right]$

$\gamma[1, i, k, l] \to \gamma a\left[\frac{e^{-t b_j}(-1+e^{-s b_j})}{b_j} + \frac{e^{-(s+t) b_j}(-1+e^{s b_j})}{b_j}, i, k, j, l\right]$

```
γ[1, k, l, m] → 0
γ[1, j, k, l] → 0
γ[1, i, j, k] → 0
a[1, j, k] → 0
a[1, j, l] → 0
a[1, i, k] → 0
a[1, i, j] → 0
a[1, k, l] → 0
a[1, l, m] → 0
γa[1, j, k, j, l] → 0
```

$$\frac{1-e^{-s b_j}}{b_j} + \frac{e^{-t b_j}\left(-1+e^{-s b_j}\right)}{b_j} + \frac{\left(-1+e^{-s b_j}\right)\left(1-e^{-t b_j}\right)}{b_j} \text{ // FullSimplify}$$

0

```
Module[{t1, t2},
   t1 = # // Ad[β[t f[bi, bj, bk]]] // Ad[β[s f[bi, bj, bk]]];
   t2 = # // Ad[β[(t + s) f[bi, bj, bk]]];
   # → t1 - t2 // S] & /@ {a[g[bi, bj, bk], i, j]} // ColumnForm
```

a[g[b_i, b_j, b_k], i, j] → 0

R

```
Switch[7,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t bi, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[γ[t, i, j], Ad[a[1, i, j]][x]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[bi, bj]]],
  4, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bj] + bi f1[bj]]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bj] + bi f1[bj]]] // Ad[γ[g, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[bj] + bi f1[bj]]] // Ad[γ[g[bi, bj], i, j]] //
     Ad[γa[-(e^-bi (2 - 2 e^bi + bi + e^bi bi))/(2 bi^3), i, j, i, j]],
  7, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[γa[-(e^-bi (2 - 2 e^bi + bi + e^bi bi))/(2 bi^3), i, j, i, j]]
 ];
VerifyR3[expr_] := Module[{lhs, rhs}, {
   lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // S;
   rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // S;
   S[lhs - rhs] == 0
  }]
```

$$-\frac{e^{-b_i}\left(2-2e^{b_i}+b_i+e^{b_i}b_i\right)}{2b_i^3}$$

Verifying R3

```
VerifyR3 /@ {γ[f, 1, 4], γ[f, 2, 4], γ[f, 3, 4], γ[f, 4, 1], γ[f, 4, 2], γ[f, 4, 3]}
```

{{True}, {True}, {True}, {True}, {True}, {True}}

```
VerifyR3 /@ {a[f[b1, b2, b3, b4], 1, 4], a[f, 2, 4]}
```

{{True}, {True}}

```
VerifyR3 /@ {a[f[b1, b2, b3, b4], 1, 4], a[f[b1, b2, b3, b4], 2, 4], a[f[b1, b2, b3, b4], 3, 4]}
```

$$\left\{\{\text{True}\}, \{\text{True}\}, \left\{\gamma\left[\frac{\left(-1+e^{b_1}\right) f[b_1, b_2, b_3, b_4] \left(2-2e^{b_2}+\left(1+e^{b_2}\right) b_2\right) b_3}{2 b_1 b_2}, 1, 2, 4\right] + \gamma a\left[-\frac{\left(-1+e^{b_1}\right) f[b_1, b_2, b_3, b_4] \left(2-2e^{b_2}+\left(1+e^{b_2}\right) b_2\right) b_3}{2 b_1 b_2^2}, 1, 2, 2, 4\right] == 0\right\}\right\}$$