

Deriving Gassner. \mathcal{L}^{2Dw} is $\mathbb{Q}[[b_i]]\langle a_{ij} \rangle$ modulo locality, $[a_{ij}, a_{ik}] = 0$, $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_j a_{ik} - b_i a_{jk}$, and $(\text{mod } \langle a_{ii} \rangle) [a_{ij}, a_{ji}] = b_i a_{ji} - b_j a_{ij}$. Acts on $\mathbf{V} = \mathbb{Q}[[b_i]]\langle x_i = a_{i\infty} \rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_i x_j - b_j x_i$. Hence $e^{\text{ad } a_{ij}} x_i = x_i$, $e^{\text{ad } a_{ij}} x_j = e^{b_i} x_j + \frac{b_j}{b_i} (1 - e^{b_i}) x_i$. Renaming $\bar{x}_i = x_i/b_i$, $\bar{t}_i = e^{b_i}$, get $[e^{\text{ad } a_{ij}}]_{\bar{x}_i, \bar{x}_j} = \begin{pmatrix} 1 & 1 - \bar{t}_i \\ 0 & \bar{t}_i \end{pmatrix}$.

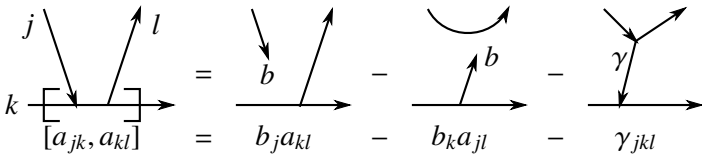
The \mathcal{L}^{2Dw} Adjoint representation. $e^{\text{ad } a_{ij}}$ acts by $a_{kl} \mapsto a_{kl}$, $a_{ik} \mapsto a_{ik}$, $a_{kj} \mapsto e^{-b_i} a_{kj} + \frac{b_k}{b_i} (1 - e^{-b_i}) a_{ij}$, $a_{ki} \mapsto a_{ki} + (1 - e^{-b_i}) a_{kj} + b_k \frac{e^{-b_i} - 1}{b_i} a_{ij}$, $a_{jk} \mapsto e^{b_i} a_{jk} + \frac{b_j}{b_i} (1 - e^{b_i}) a_{ik}$, $a_{ji} \mapsto e^{b_i} a_{ji} + \frac{b_j}{b_i} (1 - e^{b_i}) a_{ij}$.

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf.

Adjoint Gassner. Renaming $\bar{a}_{ij} = a_{ij}/b_i$ and $\bar{t}_i = e^{b_i}$, get $[\bar{a}_{ij}, \bar{a}_{ik}] = 0$, $[\bar{a}_{ik}, \bar{a}_{jk}] = -[\bar{a}_{ij}, \bar{a}_{jk}] = \bar{a}_{ik} - \bar{a}_{jk}$, and $(\text{mod } \langle \bar{a}_{ii} \rangle) [\bar{a}_{ij}, \bar{a}_{ji}] = \bar{a}_{ji} - \bar{a}_{ij}$, so

$$\begin{aligned} \bar{a}_{kj} &\mapsto \bar{t}_i^{-1} \bar{a}_{kj} + (1 - \bar{t}_i^{-1}) \bar{a}_{ij}, \\ \bar{a}_{ki} &\mapsto \bar{a}_{ki} + (1 - \bar{t}_i^{-1}) \bar{a}_{kj} + (\bar{t}_i^{-1} - 1) \bar{a}_{ij}, \\ \bar{a}_{jk} &\mapsto \bar{t}_i \bar{a}_{jk} + (1 - \bar{t}_i) \bar{a}_{ik}, \quad \bar{a}_{ji} \mapsto \bar{t}_i \bar{a}_{ji} + (1 - \bar{t}_i) \bar{a}_{ij}. \end{aligned}$$

Questions. • As Gassner is Γ calculus, Adjoint Gassner must factor through Gassner. **How?** • Interpretation? π_T -Artin?



2Dv. b : bracket trace; c : cobracket trace; $\langle b, c \rangle = \delta \in \{0, 1\}$; $\text{deg } b_i = \text{deg } c_j = \text{deg } a_{ij} = \text{deg } \delta = 1$.

\mathcal{A}^{2Dv} is $\mathbb{Q}[[\delta]]FA(b_i, c_j, a_{ij})$ (so $\mathcal{L}^v = \{f + f^{ij} a_{ij}\}$) modulo locality,

tt. (note $\gamma_{jkl} = 0$) $[a_{jk}, a_{jl}] = c_l a_{jk} - c_k a_{jl} =: \gamma_{jkl}$,

hh. $[a_{jk}, a_{ik}] = b_i a_{jk} - b_j a_{ik}$,

ht. $[a_{jk}, a_{kl}] = b_j a_{kl} - b_k a_{jl} - \gamma_{jkl}$,

$\Leftrightarrow [a_{ij}, a_{ji}] = ?$,

ab,ac. $[a_{ij}, b_i] = -[a_{ij}, b_j] = -[a_{ij}, c_i] = [a_{ij}, c_j] = \delta a_{ij} - b_i c_j =: \gamma_{ij}$,

bc. $[b_i, c_j] = 0$.

So $a_{ij} f = f^\delta a_{ij} - \frac{b_i c_j}{\delta} (f^\delta - f)$, $[a_{ij}, f] = (f^\delta - f) \left(a_{ij} - \frac{b_i c_j}{\delta} \right)$,

with $f^\delta := f \left(\frac{b_i \rightarrow b_i + \delta \quad b_j \rightarrow b_j - \delta}{c_i \rightarrow c_i - \delta \quad c_j \rightarrow c_j + \delta} \right)$.

Repeating indices. In a_{ii} , γ_{ii} , γ_{iij} , γ_{ijj} , and γ_{iii} , heads above tails.

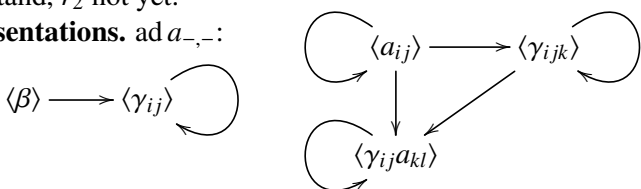
The Ascending Algebra \mathcal{A}_+^{2Dv} . Same but with only a_{ij} , $i < j$.

The OneCo Sub-Quotient is $\langle a_{ij} \rangle$ modulo $\delta^2 = \delta c_i = c_j c_k = 0$, so \mathcal{L}^{co} is (coefficient functions non-central, in $\mathbb{Q}[[b_i]]$)

$$\left\{ f^{ij} a_{ij} + f^{jk} \gamma_{ijk} + f^{jkl} \gamma_{ija_{kl}} \right\} \left| \begin{array}{l} r_1: b_i \gamma_{ijk} = \gamma_{ija_{ik}} - \gamma_{ika_{ij}} \\ r_2: b_i \gamma_{jkl} = \gamma_{jka_{il}} - \gamma_{jla_{ik}} \end{array} \right. \quad (\text{is there a residual } 4T/6T?)$$

Comment. $b_3 \gamma_{124} - b_1 \gamma_{324} = \gamma_{12} a_{34} + \gamma_{34} a_{12} - \gamma_{14} a_{32} - \gamma_{32} a_{14}$ I understand, r_2 not yet.

Representations. $\text{ad } a_{-,-}:$



• $\langle \gamma_{ij} \rangle$ is ...

OneCo $\text{ad}(a_{jk})$: (authorities: pensieve://2015-06/)

- $a[1, ij] \rightarrow a[-b_i, jk] + a[b_j, ik] + \gamma[1, ijk]$
- $a[1, ik] \rightarrow a[b_i, jk] + a[-b_j, ik]$
- $a[1, j1] \rightarrow \gamma[1, jk1]$
- $a[1, kl] \rightarrow a[b_j, kl] + a[-b_k, j1] + \gamma[-1, jkl]$
- $\beta[f[b_j, b_k]] \rightarrow \gamma[-f^{(0,1)}[b_j, b_k] + f^{(1,0)}[b_j, b_k], jk]$
- $\gamma[1, ij] \rightarrow \gamma[b_j, ik]$
- $\gamma[1, ik] \rightarrow \gamma[-b_j, ik]$
- $\gamma[1, jk] \rightarrow \gamma[-b_j, jk]$
- $\gamma[1, kl] \rightarrow \gamma[b_j, kl] + \gamma[-b_k, j1]$
- $\gamma[1, ijk] \rightarrow \gamma[-b_j, ijk] + \gamma a[1, ijjk] + \gamma a[1, ikjk]$
- $\gamma[1, ij1] \rightarrow \gamma a[1, ikj1]$
- $\gamma[1, ik1] \rightarrow \gamma a[-1, ikj1]$
- $\gamma[1, jk1] \rightarrow \gamma a[-1, jkj1]$
- $\gamma[1, klm] \rightarrow \gamma[b_j, klm] + \gamma[-b_k, jlm]$
- $\gamma a[1, jkj1] \rightarrow \gamma a[-b_j, jkj1]$

OneCo $\text{Ad}(a_{jk})$:

$$\begin{aligned} a[1, ij] &\rightarrow a[1, ij] + a[1 - e^{-t b_j}, ik] + a\left[\frac{(1 - e^{-t b_j}) b_i}{b_j}, jk\right] + \\ &\gamma\left[\frac{1 - e^{-t b_j}}{b_j}, ijk\right] + \gamma a\left[\frac{-1 + e^{-t b_j} + t b_i}{b_j^2}, ijjk\right] + \gamma a\left[\frac{-1 + e^{-t b_j} + t b_i}{b_j^2}, ikjk\right] + \\ &\gamma a\left[\frac{b_i(1 - e^{-2t b_j} - 2e^{-t b_j} t b_j)}{b_j^3}, jkj k\right] + \gamma a\left[\frac{e^{-2t b_j}(1 + e^{t b_j}(-1 + t b_i))}{b_j^3}, jkik\right] \end{aligned}$$

$$\begin{aligned} a[1, ik] &\rightarrow a[e^{-t b_j}, ik] + a\left[\frac{(1 - e^{-t b_j}) b_i}{b_j}, jk\right] + \\ &\gamma a\left[\frac{e^{-2t b_j} b_i(1 - e^{-2t b_j} + 2e^{t b_j} t b_j)}{b_j^3}, jkj k\right] + \\ &\gamma a\left[\frac{e^{-2t b_j}(-1 + e^{t b_j}(1 - t b_i))}{b_j^3}, jkik\right] \end{aligned}$$

$$a[1, j1] \rightarrow a[1, j1] + \gamma[t, jk1] + \gamma a\left[\frac{1 - e^{-t b_j} - t b_i}{b_j^2}, jkj1\right]$$

$$\begin{aligned} a[1, kl] &\rightarrow a[e^{t b_j}, kl] + a\left[\frac{(1 - e^{t b_j}) b_k}{b_j}, j1\right] + \\ &\gamma\left[\frac{t b_j b_k + (1 - e^{t b_j})(b_j + b_k)}{b_j^2}, jk1\right] + \gamma a\left[\frac{1 + e^{t b_j}(-1 + t b_i)}{b_j^2}, jkk1\right] + \\ &\gamma a\left[\frac{-2b_j + e^{t b_j} b_j - 2b_k - t b_j b_k + e^{t b_j}(b_j + 2b_k - t b_j b_k)}{b_j^3}, jkj1\right] \end{aligned}$$

$$\begin{aligned} \beta[f[b_j, b_k]] &\rightarrow \\ &\gamma\left[\frac{(-1 + e^{-t b_j})(f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k])}{b_j}, jk\right] + \beta[f[b_j, b_k]] \end{aligned}$$

- $\gamma[1, ij] \rightarrow \gamma[1, ij] + \gamma[1 - e^{-t b_j}, ik]$
- $\gamma[1, ik] \rightarrow \gamma[e^{-t b_j}, ik]$
- $\gamma[1, jk] \rightarrow \gamma[e^{-t b_j}, jk]$
- $\gamma[1, kl] \rightarrow \gamma[e^{t b_j}, kl] + \gamma\left[\frac{(1 - e^{t b_j}) b_k}{b_j}, j1\right]$
- $\gamma[1, ijk] \rightarrow \gamma[e^{-t b_j}, ijk] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, ijjk\right] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, ikjk\right]$
- $\gamma[1, ij1] \rightarrow \gamma[1, ij1] + \gamma[1 - e^{-t b_j}, ik1] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, iljk\right]$
- $\gamma[1, ik1] \rightarrow \gamma[e^{-t b_j}, ik1] + \gamma a\left[\frac{-1 + e^{-t b_j}}{b_j}, iljk\right]$
- $\gamma[1, jk1] \rightarrow \gamma[1, jk1] + \gamma a\left[\frac{-1 + e^{-t b_j}}{b_j}, jkj1\right]$
- $\gamma[1, klm] \rightarrow \gamma[e^{t b_j}, klm] + \gamma\left[\frac{(1 - e^{t b_j}) b_k}{b_j}, j1m\right]$
- $\gamma a[1, jkj1] \rightarrow \gamma a[e^{-t b_j}, jkj1]$

To do. • Perhaps I should find a way to highlight the fact that v is a perturbation of w . • Position FiC. • Position the 2D Lie bialgebras. • Is there a meaningful $a_{ij} \rightarrow a_{ij}/b_i$ (etc) renormalization? • Resolve the IHX issue! • Add: diagrammatic interpretations of $b_i, c_j, \gamma_{ij}, \gamma_{ijk}$.

Recycling.

Models. • In $[x, y] = \delta x$, $xf(y) = f(y + \delta)x$. If $\delta^2 = 0$, $[x, f(y)] = \delta f'(y)x$.

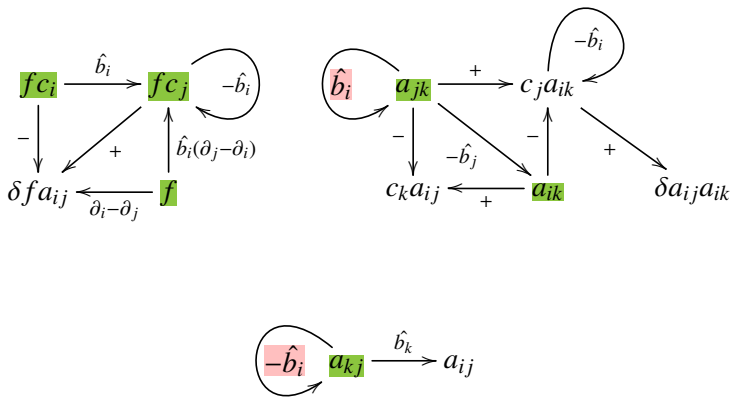
• In $[x, y] = \delta x + z^2$, $xf(y) = f(y + \delta)x + \frac{z^2}{\delta}(f(y + \delta) - f(y))$. If $\delta^2 = 0$, $[x, f(y)] = \delta f'(y)x + z^2 f'(y)$.

• If $S_n := \sum_{k=0}^{n-1} A^k C B^{n-1-k}$ then $AS_n - S_n B = A^n C - C B^n$ so $S_n = (L_A - R_B)^{-1}(A^n C - C B^n)$.

• If $\psi(x) = \sum_{n \geq 0} a_n x^n$ then $\sum_{n \geq 0} a_n \sum_{k=0}^{n-1} b^n (-b)^{n-1-k} = (\psi(b) - \psi(-b))/2b$.

The primitivity condition. $\ker(f + f^{ij} a_{ij} \mapsto \delta f + f^{ij} b_i c_j)$. (Ignoring multiple arrows).

State Diagrams. $\text{ad } a_{ij}$ yields green: roots. pink: wrong.



so with $\phi_0 := \phi(0)$, $\phi_1 := \phi'_0$, and $\phi_{\downarrow}(x) := (\phi(x) - \phi_0)/x$, $\phi(\text{ad } a_{ij})$ is

$$fc_i \mapsto \phi_0 fc_i + (b_i \phi_{\downarrow}(-b_i) - \phi_1) \delta f a_{ij} + b_i \phi_{\downarrow}(-b_i) fc_j$$

$$fc_j \mapsto \phi(-b_i) fc_j + \phi_{\downarrow}(-b_i) \delta f a_{ij}$$

$$f \mapsto \phi_0 f + b_i \phi_{\downarrow}(-b_i) (\partial_j f - \partial_i f) c_j$$

$$+ (b_i \phi_{\downarrow}(-b_i) - \phi_1) (\partial_j f - \partial_i f) \delta a_{ij}$$

$\delta a_{..} \mapsto$ as in Adjoint Gassner

$$a_{ik} \mapsto \phi_0 a_{ik} + \phi_1 c_k a_{ij} - \phi_{\downarrow}(-b_i) c_j a_{ik} - \phi_{\downarrow}(-b_i) \delta a_{ij} a_{ik}$$

$$a_{jk} \mapsto \phi(b_i) a_{jk} - (\phi_{\downarrow}(b_i) + b_j \phi_{\downarrow}(b_i)) c_k a_{ij} - b_j \phi_{\downarrow}(b_i) a_{ik}$$

$$+ \frac{\phi(b_i) - \phi(-b_i) + b_j (\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i))}{2b_i} c_j a_{ik}$$

$$+ \frac{\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i) + b_j (\phi_{\downarrow}(b_i) - \phi_{\downarrow}(-b_i))}{2b_i} \delta a_{ij} a_{ik}$$

$$a_{kj} \mapsto$$

$$a_{ij} \mapsto a_{ij}$$

Then $[a_{ij}, f] = (\partial_i f - \partial_j f) \gamma_{ij}$ and

$$\gamma_{\mathbf{b}}. \quad [\gamma_{ij}, b_l] = 0 \text{ and } [\gamma_{ijk}, b_l] = 0 \quad \text{incl. } l \in \{i, j, k\},$$

$$\mathbf{tt}\gamma. \quad [a_{jk}, \gamma_{jl}] = 0,$$

$$\mathbf{hh}\gamma. \quad [a_{jk}, \gamma_{ik}] = -b_j \gamma_{ik},$$

$$\mathbf{th}\gamma. \quad [a_{jk}, \gamma_{ij}] = b_j \gamma_{ik},$$

$$\mathbf{ht}\gamma. \quad [a_{jk}, \gamma_{kl}] = b_j \gamma_{kl} - b_k \gamma_{jl},$$

$$\mathbf{tt}\gamma_3. \quad [a_{jk}, \gamma_{jlm}] = 0,$$

$$\mathbf{th}\gamma_3. \quad [a_{jk}, \gamma_{ijl}] = b_j \gamma_{ikl} + \gamma_{il} a_{jk},$$

$$\mathbf{ht}\gamma_3. \quad [a_{jk}, \gamma_{klm}] = b_k \gamma_{jkl} + b_j \gamma_{klm},$$

$$\mathbf{hh}\gamma_3. \quad [a_{jk}, \gamma_{nik}] = -b_j \gamma_{nik} + \gamma_{ni} a_{jk},$$

$$[a_{jk}, \gamma_{jkl}] = -\gamma_{jk} a_{jl},$$

$$[a_{jk}, \gamma_{ijk}] = -b_j \gamma_{ijk} + \gamma_{ij} a_{jk} + \gamma_{ik} a_{jk}.$$