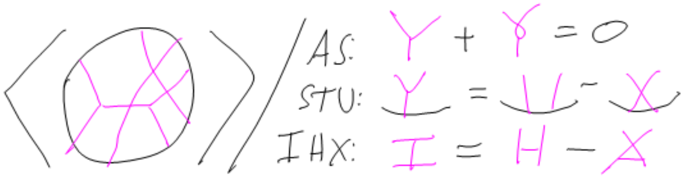
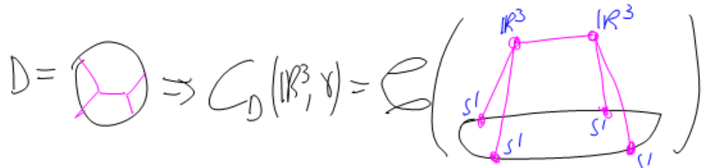
Day 3: Chern-Simons, Gaussian Integration, Feynman Diagrams

Recall. $\mathcal{K} = \{\text{knots}\}$, $\mathcal{A} := \operatorname{gr}\mathcal{A} = \mathcal{D}/\text{rels} =$



Seek $Z\colon \mathcal{K} \to \hat{\mathcal{A}}$ such that if K is n-singular, $Z(K) = D_k + \dots$

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a ``Lie'' algebra } \mathfrak{g}} \text{``}\mathcal{U}(\mathfrak{g})\text{''}$$



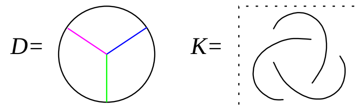
Theorem. Given a parametrized knot γ in $\mathbb{R}^3$, up to renormalizing the "framing anomaly",

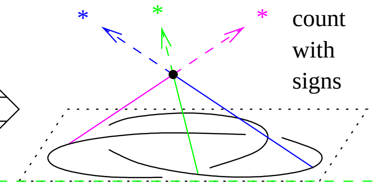
$$Z(\gamma) = \sum_{D\in\mathcal{D}} \frac{C(D)D}{|\operatorname{Aut}(D)|} \int_{C_D(\mathbb{R}^3,\gamma)} \bigwedge_{e\in E(D)} \phi_e^*\omega \in \mathcal{A}$$

is an expansion. Here $\mathcal{D}$ is the set of all "Feynman diagrams", $E(D)$ is the set of internal edges (and chords) of D, $C_D(\mathbb{R}^3,\gamma)$ is the configuration space of placements of D on/around γ, $\phi\colon C_D(\mathbb{R}^3,\gamma) \to (S^2)^{E(D)}$ is the "direction of the edges" map, and ω is a volume form on S^2.

Cosmic Coincidences

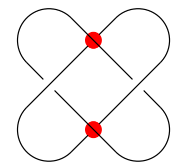
$\langle D, K\rangle_{\overline{\text{m}}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$:





The Gaussian linking number

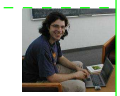
$$lk(\ ) = \sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs}) = \langle\ ,\ \rangle_{\overline{\text{m}}}$$



C.F. Gauss

The generating function of all cosmic coincidences:

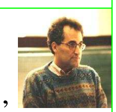
$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{\langle D, K\rangle_{\overline{\text{m}}} D}{2^c c! \binom{N}{e}} \in \mathcal{A}$$



D. Thurston

Claim. It all comes from the Chern-Simons-Witten theory,

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, \operatorname{tr}_R hol_\gamma(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right],$$



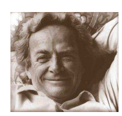
E. Witten

where $\Omega^1(\mathbb{R}^3,\mathfrak{g})$ is the space of all $\mathfrak{g}$-valued 1-forms on $\mathbb{R}^3$ (really, connections), k is some large constant, R is some representation of $\mathfrak{g}$ and tr_R is trace in R, and $hol_\gamma(A)$ is the holonomy of A along γ.

References. Witten's *Quantum field theory and the Jones polynomial*, Axelrod-Singer's *Chern-Simons perturbation theory I-II*, D. Thurston's arXiv:math.QA/9901110, Polyak's arXiv:math.GT/0406251, and my videotaped 2014 class ω/AKT.

Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

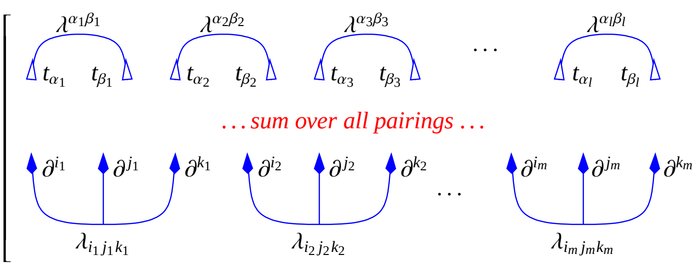
$$\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j + \frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac12\lambda_{ij}x^ix^j}$$



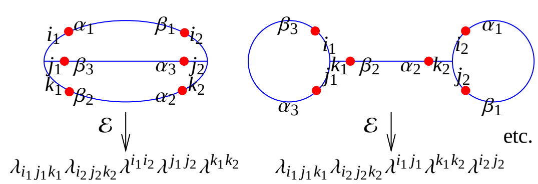
Feynman

$$= \sum_{m\geq0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m! 2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m! 2^l l!}$$



$$= \sum_{\substack{m,l\geq0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m! 2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$



$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\operatorname{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m! 2^l l!}{|\operatorname{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m] \times [(S_2)^l \rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\operatorname{Aut}(D)$. $\square$

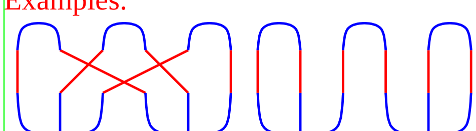
The Fourier Transform.

$$(F\colon V\to\mathbb{C}) \Rightarrow (\tilde{f}\colon V^*\to\mathbb{C})$$

via $\tilde{F}(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde{f} \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.



$|\operatorname{Aut}(D)| = 12$ $\qquad$ $|\operatorname{Aut}(D)| = 8$

Monsters left to Slay.

- Convergence.
- Proof of invariance.
- The framing anomaly.
- Universallity.
- d^{-1} doesn't really exist, Faddeev-Popov, determinants, ghosts, Berezin integration.
- Assembly.