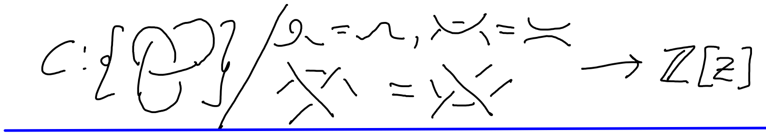
Louvain2-1

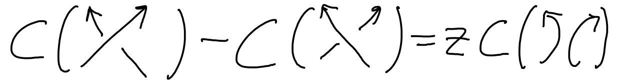
The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

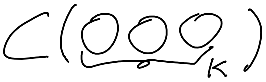
Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.



Example. The Conway polynomial is given by



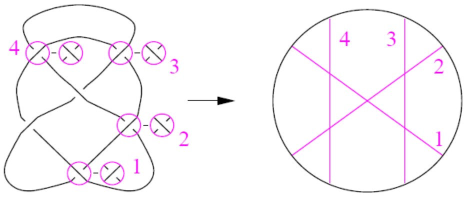
and



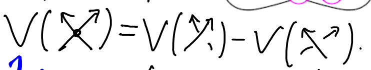
$$= \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

Definition. Any $V: \{\text{knots}\} \to \text{Abelian Group } A$ can be extended to "knots w/ double points"



using



(Think "differentiation")

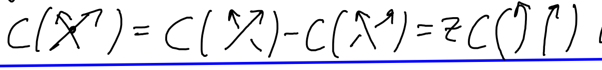
Definition. V is of type m if always

(think "polynomial")

Conjecture. Finite type invariants separate knots.

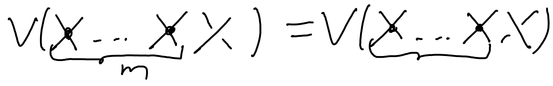
Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

Proof.

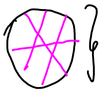


□

Let V be of type m; then $V^{(m)}$ is constant:

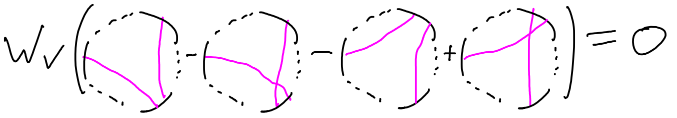


So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: $W_V : \{$

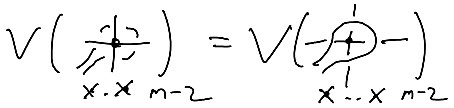


$\} \longrightarrow A$

Claim. W_V satisfies the 4T relation:



Proof.



□

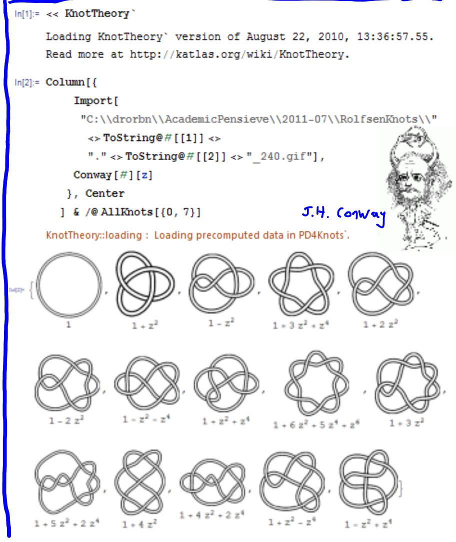
Exercise. 1. Determine the "weight system" W_m of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.
2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W\ \exists V$ s.t. $W = W_V$

M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}_m^r$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Proposition. The Fundamental thm holds iff there exists an expansion: $Z: \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.



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In[1]:= << KnotTheory`

        Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
        Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
          Import[
           "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
            <> ToString@#[[1]] <>
            "." <> ToString@#[[2]] <> "_240.gif"],
          Conway[#][z]
         }, Center
        ] & /@ AllKnots[{0, 7}]

        KnotTheory::loading :  Loading precomputed data in PD4Knots`.

Out[2]= {  1,  1 + z^2,  1 - z^2,  1 + 3 z^2 + z^4,  1 + 2 z^2,
           1 - 2 z^2,  1 - z^2 - z^4,  1 + z^2 + z^4,  1 + 6 z^2 + 5 z^4 + z^6,  1 + 3 z^2,
           1 + 5 z^2 + 2 z^4,  1 + 4 z^2,  1 + 4 z^2 + 2 z^4,  1 + z^2 - z^4,  1 - z^2 + z^4 }
```

J.H. Conway

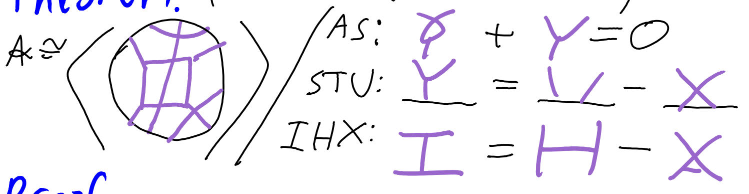
Proof.

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[dr, "V"'] & \mathcal{A} \arrow[d, "W"] \\ & \mathbb{Q} \end{tikzcd}$$

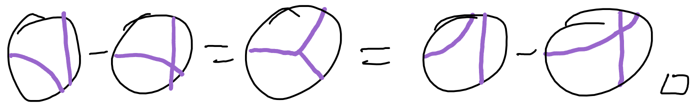
□

Note. Z is precisely an "Expansion" in the sense of yesterday.

Theorem. (The "bracket-rise" theorem).



Proof



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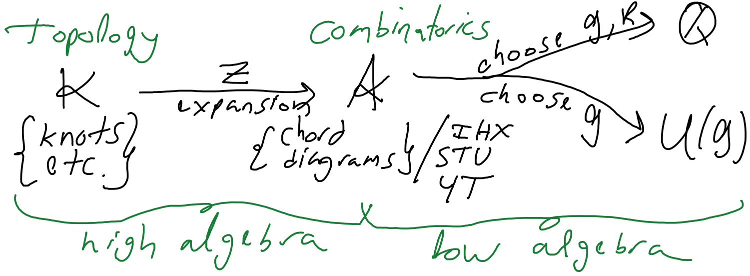
Also see my old paper, "On the Vassiliev Knot Invariants" (a web search will find...)

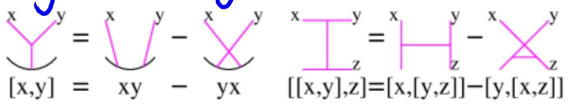
Low and High Algebra in the "u" Case

Dror Bar-Natan in Louvain-la-Neuve, June 2015, http://www.math.toronto.edu/~drorbn/Talks/Louvain-1506

The big picture, "u" Case.



Very low algebra.



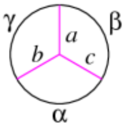
[x,y] = xy − yx   [[x,y],z]=[x,[y,z]]−[y,[x,z]]

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [X_a, X_c], X_c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}:$



$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

Universal Representation Theory.

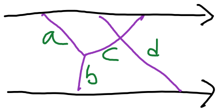
Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal{U}(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

* Every rep of $\mathfrak{g}$ extends to $\mathcal{U}(\mathfrak{g})$.

* $\exists\, \Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y \rangle / [x,y] = x$, determine $\mathcal{U}(\mathfrak{g})$. Guess a generalization.

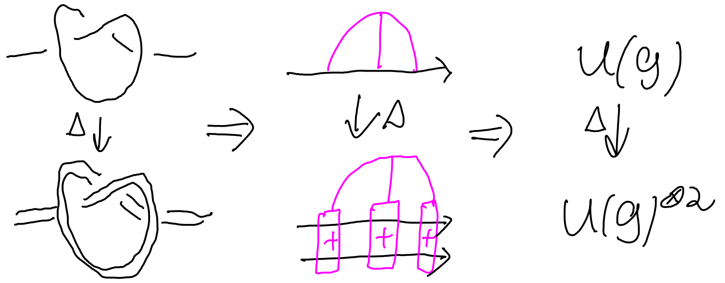
Low algebra. $\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via



$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

& likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n} \Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

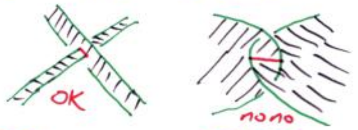


A "Homomorphic Expansion" $Z : \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is High Algebra.

$$\{\text{Ribbon knots}\} \underset{\text{more or less}}{=} \left\{ u\gamma : \begin{array}{l} \gamma \in \mathcal{K}(\text{O-O}) \\ d\gamma = \text{OO} \end{array} \right\}$$

where Ribbon means



and

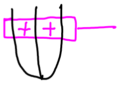
Algebraic knot theory:

$$\begin{tikzcd}
 & \mathcal{K}(\text{OO}) \arrow[r, "Z"] & \mathcal{A}(\text{OO}) \ni \text{OO} \\
\mathcal{K}(\text{O-O}) \arrow[ur, "d"] \arrow[rr, "Z"] \arrow[dr, "u"'] & & \mathcal{A}(\text{O-O}) \arrow[u, "d"'] \arrow[d, "u"] \\
 & \mathcal{K}(\text{O=O}) \arrow[r, "Z"] & \mathcal{A}(\text{O})
\end{tikzcd}$$

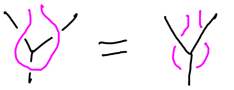
So

$$Z\left(\left\{\begin{array}{c}\text{Ribbon}\\ \text{knots}\end{array}\right\}\right) \subset \{u\alpha : d\alpha = Z(\text{OO})\} \subset \mathcal{A}(\text{O-O})$$

VI



= 0, follows from



Talk II: From Knots to Lie Algebras. Why on Earth should knots be related to Lie algebras? The former are squishy and irregular, the latter are symmetric and rigid. They should know nothing of each other. Yet as we shall see, the natural target space for expansions for knots is in some sense, "the universal dual" of all (metrized) Lie algebras.