Pensieve header: One-Co computations; continues pensieve://2015-04/, continued pensieve://2015-06/.

The bracket

In the basis {a[f,i,j], γ[f,i,j], γ[f,i,j,k], γa[f,i,j,k,l]}.

Generalities

```
DQ[is___] := (Sort[{is}] === Union[{is}]);
```

```
Simp[expr_] := Simplify[expr];
LSimp[γ[f_, j_, k_, k_]] = 0;
LSimp[γ[f_, j_, k_, l_]] /; OrderedQ[{l, k}] ∧ DQ[k, l] := γ[-f, j, l, k] // LSimp;
LSimp[γa[f_, i_, j_, i_, k_]] /; OrderedQ[{k, j}] ∧ DQ[k, j] :=
  γ[b_i f, i, j, k] + γa[f, i, k, i, j] // LSimp;
LSimp[β[f_]] := β[Simp[f]];
LSimp[a[i_, j_]] := a[i, j];
LSimp[a[f_, i_, j_]] := a[Simp[f], i, j];
LSimp[γ[f_, is__]] := γ[Simp[f], is];
LSimp[γa[f_, is__]] := γa[Simp[f], is];
LSimp[expr_] := expr /. (λ_β | λ_a | λ_γ | λ_γa) ⧴ LSimp[λ];
```

```
β[0] := 0;
β /: β[f_] + β[g_] := β[f + g];
β /: -β[f_] := β[-f] // LSimp;
a[0, __] := 0;
a /: a[f_, i_, j_] + a[g_, i_, j_] := a[f + g, i, j];
a /: -a[f_, is__] := a[-f, is] // LSimp;
γ[0, __] := 0;
γ /: γ[f_, is__] + γ[g_, is__] := γ[f + g, is];
γ /: -γ[f_, is__] := γ[-f, is] // LSimp;
γa[0, __] := 0;
γa /: γa[f_, is__] + γa[g_, is__] := γa[f + g, is];
γa /: -γa[f_, is__] := γa[-f, is] // LSimp;
```

```
B[0, _] = 0; B[_, 0] = 0;
B[x_, x_] = 0;
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
```

Specific Brackets

[*a*, *β*] brackets:

OneCoBrackets

```
B[a[f_, j_, k_], β[g_]] := γ[f (∂_{b_j} g - ∂_{b_k} g), j, k] // LSimp;
```

[a,a] brackets:

```
B[a[f_, j_, k_], a[g_, l_, m_]] := Plus[
    B[a[j, k], a[l, m]] /. {a[h_, is__] ⧴ a[f g h, is], γ[h_, is__] ⧴ γ[f g h, is]},
    γa[f (∂_{b_j} g - ∂_{b_k} g), j, k, l, m] + γa[g (∂_{b_m} f - ∂_{b_l} f), l, m, j, k]
   ] // LSimp;
```

OneCoBrackets

```
B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // LSimp;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] := a[b_i, j, k] - a[b_j, i, k] // LSimp;
B[a[j_, k_], a[i_, j_]] /; DQ[i, j, k] := a[b_j, i, k] - a[b_i, j, k] + γ[1, i, j, k] // LSimp;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := -B[a[k, l], a[j, k]];
B[a[j_, k_], a[l_, m_]] /; DQ[j, k, l, m] := 0;
```

[a, γ_2] brackets: (only γ_2's in output!)

OneCoBrackets

```
B[a[f_, j_, k_], γ[g_, j_, l_]] /; DQ[j, k, l] := 0;
B[a[f_, j_, k_], γ[g_, i_, k_]] /; DQ[i, j, k] := - γ[b_j f g, i, k] // LSimp;
B[a[f_, j_, k_], γ[g_, i_, j_]] /; DQ[i, j, k] := γ[b_j f g, i, k] // LSimp;
B[a[f_, j_, k_], γ[g_, k_, l_]] /; DQ[j, k, l] := γ[b_j f g, k, l] - γ[b_k f g, j, l] // LSimp;
B[a[f_, j_, k_], γ[g_, l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, j_, k_]] := γ[-b_j f g, j, k] // LSimp;
```

[a, γ_3] brackets:

OneCoBrackets

```
B[a[f_, j_, k_], γ[g_, j_, l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, i_, j_, l_]] /; DQ[i, j, k, l] :=
  γ[b_j f g, i, k, l] + γa[f g, i, l, j, k] // LSimp;
B[a[f_, l_, k_], γ[g_, i_, j_, l_]] /; DQ[i, j, k, l] :=
  γ[-b_l f g, i, k, j] + γa[-f g, i, j, l, k] // LSimp;
B[a[f_, j_, k_], γ[g_, k_, l_, m_]] /; DQ[j, k, l, m] :=
  γ[-b_k f g, j, l, m] + γ[b_j f g, k, l, m] // LSimp;
B[a[f_, j_, k_], γ[g_, n_, i_, k_]] /; DQ[n, i, j, k] :=
  γ[-b_j f g, n, i, k] + γa[f g, n, i, j, k] // LSimp;
B[a[f_, j_, i_], γ[g_, n_, i_, k_]] /; DQ[n, i, j, k] :=
  γ[b_j f g, n, k, i] + γa[-f g, n, k, j, i] // LSimp;
B[a[f_, j_, k_], γ[g_, j_, k_, l_]] /; DQ[j, k, l] := γa[-f g, j, k, j, l] // LSimp;
B[a[f_, j_, l_], γ[g_, j_, k_, l_]] /; DQ[j, k, l] := γa[f g, j, l, j, k] // LSimp;
B[a[f_, j_, k_], γ[g_, i_, j_, k_]] /; DQ[i, j, k] :=
  γ[-b_j f g, i, j, k] + γa[f g, i, j, j, k] + γa[f g, i, k, j, k] // LSimp;
B[a[f_, k_, j_], γ[g_, i_, j_, k_]] /; DQ[i, j, k] :=
  γ[b_k f g, i, k, j] + γa[-f g, i, k, k, j] + γa[-f g, i, j, k, j] // LSimp;
B[a[f_, i_, j_], γ[g_, k_, l_, m_]] /; DQ[i, j, k, l, m] := 0;
```

[a, γa] brackets:

```
B[x_a, γa[f_, i_, j_, m_, n_]] := Plus[
    B[x, γ[f, i, j]] /. γ[g_, k_, l_] ⧴ γa[g, k, l, m, n],
    B[x, a[1, m, n]] /. {a[g_, k_, l_] ⧴ γa[f g, i, j, k, l], _γ | _γa ⧴ 0}
  ] // LSimp
```

[β,a], [γ, a], [γ,γ] brackets:

```
B[x_β | x_γ | x_γa, y_a] := -B[y, x];
B[_β | _γ | _γa, _β | _γ | _γa] := 0;
```

Testing Jacobi and Anti-Symmetry

```
FormalPlusBasis[n_, f_] := Module[{ff},
    ff = f @@ Table[b_i, {i, n}];
    Flatten@{
        Table[a[ff, i, j], {i, n - 1}, {j, i + 1, n}],
        Table[γ[ff, i, j], {i, n - 1}, {j, i + 1, n}],
        Table[γ[ff, i, j, k], {i, n - 2}, {j, i + 1, n - 1}, {k, j + 1, n}],
        Table[γa[ff, i, j, k, l], {i, n - 1}, {j, i + 1, n}, {k, n - 1}, {l, k + 1, n}]
      } /. 1[___] → 1
  ];
```

```
FormalPlusBasis[3, f]
```

```
{a[f[b1, b2, b3], 1, 2], a[f[b1, b2, b3], 1, 3], a[f[b1, b2, b3], 2, 3],
 γ[f[b1, b2, b3], 1, 2], γ[f[b1, b2, b3], 1, 3], γ[f[b1, b2, b3], 2, 3], γ[f[b1, b2, b3], 1, 2, 3],
 γa[f[b1, b2, b3], 1, 2, 1, 2], γa[f[b1, b2, b3], 1, 2, 1, 3], γa[f[b1, b2, b3], 1, 2, 2, 3],
 γa[f[b1, b2, b3], 1, 3, 1, 2], γa[f[b1, b2, b3], 1, 3, 1, 3], γa[f[b1, b2, b3], 1, 3, 2, 3],
 γa[f[b1, b2, b3], 2, 3, 1, 2], γa[f[b1, b2, b3], 2, 3, 1, 3], γa[f[b1, b2, b3], 2, 3, 2, 3]}
```

```
AS[x1_, x2_] := Module[{as},
   as = B[x1, x2] + B[x2, x1] // LSimp;
   If[as === 0, as, {x1, x2} → as]
  ];
Outer[
 AS,
 FormalPlusBasis[3, f],
 FormalPlusBasis[3, g]
]
```

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

```
Jacobi[x1_, x2_, x3_] := Module[{Jac},
   Jac = LSimp[B[x1, B[x2, x3]] + B[x2, B[x3, x1]] + B[x3, B[x1, x2]]];
   If[Jac === 0, Jac, {x1, x2, x3} → Jac]
  ];
DeleteCases[
 Flatten[Outer[
   Jacobi,
   FormalPlusBasis[3, f],
   FormalPlusBasis[3, g],
   FormalPlusBasis[3, h]
  ]],
 0]
```

```
{}
```

Testing representations

The $R\langle\beta\gamma_2\rangle$ representation

```
Outer[
   Function[{x, y, r}, {x, y, r} → B[B[x, y], r] - B[x, B[y, r]] + B[y, B[x, r]]],
   {a[1, 1, 2], a[1, 1, 3], a[1, 2, 3]},
   {a[1, 1, 2], a[1, 1, 3], a[1, 2, 3]},
   {β[f[b₁, b₂, b₃]]}
  ] // Flatten // ColumnForm
```

```
{a[1, 1, 2], a[1, 1, 2], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,1,0)[b₁, b₂, b₃] - f^(1,0,0)[b₁, b₂, b₃]) + b₁ (-f^(0,1,0)[b₁, b₂,
{a[1, 1, 2], a[1, 1, 3], β[f[b₁, b₂, b₃]]} → 0
{a[1, 1, 2], a[1, 2, 3], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,0,1)[b₁, b₂, b₃] - f^(0,1,0)[b₁, b₂, b₃]) + b₁ (-f^(0,0,1)[b₁, b₂,
{a[1, 1, 3], a[1, 1, 2], β[f[b₁, b₂, b₃]]} → 0
{a[1, 1, 3], a[1, 1, 3], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,0,1)[b₁, b₂, b₃] - f^(1,0,0)[b₁, b₂, b₃]) + b₁ (-f^(0,0,1)[b₁, b₂,
{a[1, 1, 3], a[1, 2, 3], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,0,1)[b₁, b₂, b₃] - f^(0,1,0)[b₁, b₂, b₃]) + b₁ (-f^(0,0,1)[b₁, b₂,
{a[1, 2, 3], a[1, 1, 2], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,0,1)[b₁, b₂, b₃] - f^(0,1,0)[b₁, b₂, b₃]) + b₁ (-f^(0,0,1)[b₁, b₂,
{a[1, 2, 3], a[1, 1, 3], β[f[b₁, b₂, b₃]]} → γ[b₁ (f^(0,0,1)[b₁, b₂, b₃] - f^(0,1,0)[b₁, b₂, b₃]) + b₁ (-f^(0,0,1)[b₁, b₂,
{a[1, 2, 3], a[1, 2, 3], β[f[b₁, b₂, b₃]]} → γ[b₂ (f^(0,0,1)[b₁, b₂, b₃] - f^(0,1,0)[b₁, b₂, b₃]) + b₂ (-f^(0,0,1)[b₁, b₂,
```

The Adjoint action

```
AutoAd[x_][y_] := Module[{pows, states, s, seq, sh = 5, sf, t1, n},
   pows = NestList[B[x, #] &, y, 15];
   states = Union[Cases[pows, s_a | s_β | s_γ | s_γa :→ ReplacePart[s, 1 → _], ∞]];
   Sum[
    seq = Cases[{#}, s, ∞] & /@ pows;
    seq = Replace[seq, {{_[f_, ___]} :→ f, {} → 0}, {1}];
    sf = FindSequenceFunction[Drop[seq, sh]];
                                    sh-1                    ∞
                                    ___ seq⟦n + 1⟧          ___ sf[n + 1 - sh]
    ReplacePart[s, 1 → FullSimplify[ \   ──────────  +      \   ────────────── ]],
                                    /__     n !             /__      n !
                                    n=0                     n=sh
    {s, states}]
  ];
AutoAd1[x_][y_] := Module[{pows, states, s, seq, sh, sf, t1, n},
  pows = NestList[B[x, #] &, y, 15];
  states = Union[Cases[pows, s_a | s_β | s_γ | s_γa :→ ReplacePart[s, 1 → _], ∞]];
  Sum[
   seq = Cases[{#}, s, ∞] & /@ pows;
   sh = 0; While[seq⟦1⟧ === {}, ++sh; seq = Rest[seq]];
   seq = Replace[seq, {{_[f_, ___]} :→ f, {} → 0}, {1}];
   sf = Replace[seq, {
      {t1_, 0 ...} :→ (t1 * KroneckerDelta[1, #] &),
      seq_ :→ FindSequenceFunction[seq]
     }];
                                   ∞
                                  ___ sf[n + 1 - sh]
   ReplacePart[s, 1 → FullSimplify[ \   ──────────────]],
                                  /__      n !
                                  n=sh
   {s, states}
  ]
 ]
(* Hint: Perhaps improve using Variables, CoefficientList, FromCoefficientList *)
```

```
Ad[a[t_, j_, k_]][β[f_]] /; FreeQ[t, b_] := β[f] + γ[(ⅇ^(-t b_j) - 1) (∂_{b_k} f - ∂_{b_j} f)/b_j, j, k];
Ad[a[t_, j_, k_]][γ[f_, j_, k_]] /; FreeQ[t, b_] := γ[ⅇ^(-t b_j) f, j, k];
Ad[a[t_, j_, k_]][γ[f_, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] := γ[f, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := γ[ⅇ^(-t b_j) f, i, k];
Ad[a[t_, j_, k_]][γ[f_, i_, j_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := γ[f, i, j] + γ[f - ⅇ^(-t b_j) f, i, k];
Ad[a[t_, j_, k_]][γ[f_, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(t b_j) f, k, l] + γ[(1 - ⅇ^(t b_j)) f b_k/b_j, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] := γ[f, i, l];
Ad[a[t_, j_, k_]][γ[f_, j_, l_, m_]] /; DQ[j, l, k, m] ∧ FreeQ[t, b_] := γ[f, j, l, m];
Ad[a[t_, j_, k_]][γ[f_, i_, j_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] :=
  γ[f, i, j, l] + γ[f - ⅇ^(-t b_j) f, i, k, l] + γa[(f - ⅇ^(-t b_j) f)/b_j, i, l, j, k];
Ad[a[t_, j_, k_]][γ[f_, i_, k_, l_]] /; DQ[i, j, k, l] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(-t b_j) f, i, k, l] + γa[(ⅇ^(-t b_j) - 1) f/b_j, i, l, j, k];
Ad[a[t_, j_, k_]][γ[f_, k_, l_, m_]] /; DQ[j, k, l, m] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(t b_j) f, k, l, m] + γ[(1 - ⅇ^(t b_j)) f b_k/b_j, j, l, m];
Ad[a[t_, j_, k_]][γ[f_, j_, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  γ[f, j, k, l] + γa[(-1 + ⅇ^(-t b_j)) f/b_j, j, k, j, l];
Ad[a[t_, j_, k_]][γ[f_, i_, j_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
  γ[ⅇ^(-t b_j) f, i, j, k] + γa[(1 - ⅇ^(-t b_j))/b_j f, i, j, j, k] + γa[(1 - ⅇ^(-t b_j))/b_j f, i, k, j, k];
Ad[a[t_, j_, k_]][a[1, j_, k_]] /; FreeQ[t, b_] := a[1, j, k];

Ad[a[t_, j_, k_]][a[1, j_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
 a[1, j, l] + γ[t, j, k, l] + γa[(1 - ⅇ^(-t b_j) - t b_j)/b_j^2, j, k, j, l];

Ad[a[t_, j_, k_]][a[1, i_, k_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] := a[ⅇ^(-t b_j), i, k] + a[(1 - ⅇ^(-t b_j)) b_i/b_j, j, k] +
  γa[ⅇ^(-2 t b_j) b_i (2 ⅇ^(t b_j) t b_j + 1 - ⅇ^(2 t b_j))/b_j^3, j, k, j, k] + γa[ⅇ^(-2 t b_j) (ⅇ^(t b_j) (1 - t b_j) - 1)/b_j^2, j, k, i, k];
Ad[a[t_, j_, k_]][a[1, i_, j_]] /; DQ[i, j, k] ∧ FreeQ[t, b_] :=
  a[1, i, j] + a[1 - ⅇ^(-t b_j), i, k] + a[(ⅇ^(-t b_j) - 1) b_i/b_j, j, k] + γ[(1 - ⅇ^(-t b_j))/b_j, i, j, k] +
   γa[b_i (1 - 2 ⅇ^(-t b_j) t b_j - ⅇ^(-2 t b_j))/b_j^3, j, k, j, k] + γa[(ⅇ^(-t b_j) + t b_j - 1)/b_j^2, i, j, j, k] +
   γa[(ⅇ^(-t b_j) + t b_j - 1)/b_j^2, i, k, j, k] + γa[ⅇ^(-2 t b_j) (1 + ⅇ^(t b_j) (-1 + t b_j))/b_j^2, j, k, i, k];
Ad[a[t_, j_, k_]][a[1, k_, l_]] /; DQ[j, k, l] ∧ FreeQ[t, b_] :=
  a[ⅇ^(t b_j), k, l] + a[(1 - ⅇ^(t b_j)) b_k/b_j, j, l] + γ[(t b_j b_k + (1 - ⅇ^(t b_j)) (b_j + b_k))/b_j^2, j, k, l] +
   γa[(1 + ⅇ^(t b_j) (t b_j - 1))/b_j^2, j, k, k, l] + γa[(ⅇ^(-t b_j) b_j + ⅇ^(t b_j) (b_j + 2 b_k - t b_j b_k) - 2 b_j - 2 b_k - t b_j b_k)/b_j^3, j, k, j, l];
Ad[a[t_, j_, k_]][a[1, l_, m_]] /; DQ[j, k, l, m] ∧ FreeQ[t, b_] := a[1, l, m];
(*Ad[β[f_]][a[1, i_,j_]] := a[1,i,j]+γ[∂_{b_j}f-∂_{b_i}f,i,j];*)
Ad[x_β | x_γ | x_γa][y_] := y + B[x, y];
Ad[β[f_]][t_β | t_γ | t_γa] := t;
```

```
AdTests[a[t, j, k]] = {β[f[b_j, b_k]], γ[f, i, j], γ[f, i, k],
   γ[f, j, l], γ[f, k, l], γ[f, i, j, l], γ[f, j, k, l], γ[f, i, j, k], γ[f, i, k, l],
   γ[f, k, l, m], a[1, j, k], a[1, j, l], a[1, i, k], a[1, i, j], a[1, k, l]};
```

```
LSimp[AutoAd[a[t, j, k]][#] - Ad[a[t, j, k]][#]] & /@ AdTests[a[t, j, k]]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
AutoAd[β[f[b_i, b_j]]][a[g, i, j]] - Ad[β[f[b_i, b_j]]][a[g, i, j]] // LSimp
```

```
0
```

```
Ad[x_][y_Plus] := Ad[x] /@ y;
(Ad[x_][a[f_ , i_, j_]] /; f =!= 1) := Module[{Adf, g, γs},
   Adf = β[f] // Ad[x];
   Adf /. β[h_] :→ (g = h);
   γs = Cases[{Adf - β[g]}, _γ, ∞];
   LSimp[(a[1, i, j] // Ad[x]) /. {
       a[h_, k_, l_] :→ Plus[
          a[g * h, k, l],
          Total[γs /. γ[hh_, m_, n_] :→ γa[h * hh, m, n, k, l]]
         ],
       γ[h_, is__] :→ γ[g * h, is],
       γa[h_, is__] :→ γa[g * h, is]
      }]
  ];
Ad[x_][γa[f_, i_, j_, k_, l_]] := LSimp[
   Expand[(γ[f, i, j] // Ad[x]) (a[1, k, l] // Ad[x])] /. {
     γ[g_, m_, n_] a[h_, p_, q_] :→ γa[g * h, m, n, p, q],
     _γ * (_γ | _γa) → 0
    }
  ];
```

```
Ad[x_][y_] := (Print["Ad not yet defined on ", {x, y}]; y);
```

The semi group properties

```
tests = {β[f[b_j, b_k]], γ[1, i, j], γ[1, i, k], γ[1, i, l], γ[1, j, k], γ[1, j, l], γ[1, k, l],
   γ[1, j, l, m], γ[1, i, j, l], γ[1, i, k, l], γ[1, k, l, m], γ[1, j, k, l], γ[1, i, j, k],
   a[1, j, k], a[1, j, l], a[1, i, k], a[1, i, j], a[1, k, l], a[1, l, m], γa[1, j, k, j, l]};
```

OneCoAd

```
Print[# -> Ad[a[t, j, k]][#]] & /@ tests;
```

OneCoAd

$$\beta[f[b_j, b_k]] \to \beta[f[b_j, b_k]] + \gamma\left[\frac{(-1 + e^{-t b_j})\,(f^{(0,1)}[b_j, b_k] - f^{(1,0)}[b_j, b_k])}{b_j}, j, k\right]$$

OneCoAd

$$\gamma[1, i, j] \to \gamma[1, i, j] + \gamma[1 - e^{-t b_j}, i, k]$$

OneCoAd

$$\gamma[1, i, k] \to \gamma[e^{-t b_j}, i, k]$$

OneCoAd

$$\gamma[1, i, l] \to \gamma[1, i, l]$$

OneCoAd

$$\gamma[1, j, k] \to \gamma[e^{-t b_j}, j, k]$$

OneCoAd

$$\gamma[1, j, l] \to \gamma[1, j, l]$$

OneCoAd

$$\gamma[1, k, l] \to \gamma[e^{t b_j}, k, l] + \gamma\left[\frac{(1 - e^{t b_j})\, b_k}{b_j}, j, l\right]$$

OneCoAd

$$\gamma[1, j, l, m] \to \gamma[1, j, l, m]$$

OneCoAd

$$\gamma[1, i, j, l] \to \gamma[1, i, j, l] + \gamma[1 - e^{-t b_j}, i, k, l] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, i, l, j, k\right]$$

OneCoAd

$$\gamma[1, i, k, l] \to \gamma[e^{-t b_j}, i, k, l] + \gamma a\left[\frac{-1 + e^{-t b_j}}{b_j}, i, l, j, k\right]$$

OneCoAd

$$\gamma[1, k, l, m] \to \gamma[e^{t b_j}, k, l, m] + \gamma\left[\frac{(1 - e^{t b_j})\, b_k}{b_j}, j, l, m\right]$$

OneCoAd

$$\gamma[1, j, k, l] \to \gamma[1, j, k, l] + \gamma a\left[\frac{-1 + e^{-t b_j}}{b_j}, j, k, j, l\right]$$

OneCoAd

$$\gamma[1, i, j, k] \to \gamma[e^{-t b_j}, i, j, k] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, i, j, j, k\right] + \gamma a\left[\frac{1 - e^{-t b_j}}{b_j}, i, k, j, k\right]$$

OneCoAd

$$a[1, j, k] \to a[1, j, k]$$

OneCoAd

$$a[1, j, l] \to a[1, j, l] + \gamma[t, j, k, l] + \gamma a\left[\frac{1 - e^{-t b_j} - t\, b_j}{b_j^2}, j, k, j, l\right]$$

OneCoAd

$$a[1, i, k] \to a[e^{-t b_j}, i, k] + a\left[\frac{(1 - e^{-t b_j})\, b_i}{b_j}, j, k\right] + \\ \gamma a\left[\frac{e^{-2 t b_j}\, b_i\, (1 - e^{2 t b_j} + 2\, e^{t b_j}\, t\, b_j)}{b_j^3}, j, k, j, k\right] + \gamma a\left[\frac{e^{-2 t b_j}\, (-1 + e^{t b_j}\, (1 - t\, b_j))}{b_j^2}, j, k, i, k\right]$$

OneCoAd

$$a[1, i, j] \to a[1, i, j] + a[1 - e^{-t b_j}, i, k] + a\left[\frac{(-1 + e^{-t b_j})\, b_i}{b_j}, j, k\right] + \\ \gamma\left[\frac{1 - e^{-t b_j}}{b_j}, i, j, k\right] + \gamma a\left[\frac{-1 + e^{-t b_j} + t\, b_j}{b_j^2}, i, j, j, k\right] + \gamma a\left[\frac{-1 + e^{-t b_j} + t\, b_j}{b_j^2}, i, k, j, k\right] + \\ \gamma a\left[\frac{b_i\, (1 - e^{-2 t b_j} - 2\, e^{-t b_j}\, t\, b_j)}{b_j^3}, j, k, j, k\right] + \gamma a\left[\frac{e^{-2 t b_j}\, (1 + e^{t b_j}\, (-1 + t\, b_j))}{b_j^2}, j, k, i, k\right]$$

OneCoAd

$$a[1, k, l] \to a[e^{t b_j}, k, l] + a\left[\frac{(1 - e^{t b_j})\, b_k}{b_j}, j, l\right] + \gamma\left[\frac{t\, b_j\, b_k + (1 - e^{t b_j})\, (b_j + b_k)}{b_j^2}, j, k, l\right] + \\ \gamma a\left[\frac{1 + e^{t b_j}\, (-1 + t\, b_j)}{b_j^2}, j, k, k, l\right] + \gamma a\left[\frac{-2\, b_j + e^{-t b_j}\, b_j - 2\, b_k - t\, b_j\, b_k + e^{t b_j}\, (b_j + 2\, b_k - t\, b_j\, b_k)}{b_j^3}, j, k, j, l\right]$$

OneCoAd

$$a[1, l, m] \to a[1, l, m]$$

OneCoAd

$$\gamma a[1, j, k, j, l] \to \gamma a[e^{-t b_j}, j, k, j, l]$$

```
Module[{t1, t2},
  t1 = Ad[a[t, j, k]][#] /. (h : (a | β | γ | γa))[c_, r___] ⧴ h[SeriesCoefficient[c, {t, 0, 1}], r];
  t2 = B[a[1, j, k], #];
  # → t1 == t2] & /@ tests // ColumnForm
```

```
β[f[b_j, b_k]] → True
γ[1, i, j] → True
γ[1, i, k] → True
γ[1, i, l] → True
γ[1, j, k] → True
γ[1, j, l] → True
γ[1, k, l] → True
γ[1, j, l, m] → True
γ[1, i, j, l] → True
γ[1, i, k, l] → True
γ[1, k, l, m] → True
γ[1, j, k, l] → True
γ[1, i, j, k] → True
a[1, j, k] → True
a[1, j, l] → True
a[1, i, k] → True
a[1, i, j] → True
a[1, k, l] → True
a[1, l, m] → True
γa[1, j, k, j, l] → True
```

```
Module[{t1, t2},
  t1 = Ad[β[t f[b_i, b_j, b_k]]][#] /.
    (h : (a | β | γ | γa))[c_, r___] ⧴ h[SeriesCoefficient[c, {t, 0, 1}], r];
  t2 = B[β[f[b_i, b_j, b_k]], #];
  # → LSimp[t1 == t2]] & /@ {a[g[b_i, b_j, b_k], i, j]} // ColumnForm
```

```
a[g[b_i, b_j, b_k], i, j] → True
```

```
Module[{t1, t2},
  t1 = # // Ad[a[t, j, k]] // Ad[a[s, j, k]];
  t2 = # // Ad[a[t + s, j, k]];
  # → t1 - t2 // LSimp] & /@ tests // ColumnForm
```

```
β[f[b_j, b_k]] → 0
γ[1, i, j] → 0
γ[1, i, k] → 0
γ[1, i, l] → 0
γ[1, j, k] → 0
γ[1, j, l] → 0
γ[1, k, l] → 0
γ[1, j, l, m] → 0
γ[1, i, j, l] → 0
γ[1, i, k, l] → 0
γ[1, k, l, m] → 0
γ[1, j, k, l] → 0
γ[1, i, j, k] → 0
a[1, j, k] → 0
a[1, j, l] → 0
a[1, i, k] → 0
a[1, i, j] → 0
a[1, k, l] → 0
a[1, l, m] → 0
γa[1, j, k, j, l] → 0
```

```
Module[{t1, t2},
  t1 = # // Ad[β[t f[b_i, b_j, b_k]]] // Ad[β[s f[b_i, b_j, b_k]]];
  t2 = # // Ad[β[(t + s) f[b_i, b_j, b_k]]];
  # → t1 - t2 // LSimp] & /@ {a[g[b_i, b_j, b_k], i, j]} // ColumnForm
```

```
a[g[b_i, b_j, b_k], i, j] → 0
```

R

```
Switch[6,
  0, R[i_, j_][x_] := Ad[a[1, i, j]][x],
  1, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[a[t b_i, i, j], Ad[a[1, i, j]][x]],
  2, R[i_, j_][x_] := Ad[a[1, i, j]][x] + B[γ[t, i, j], Ad[a[1, i, j]][x]],
  3, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[r[b_i, b_j]]],
  4, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[b_j] + b_i f1[b_j]]],
  5, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[b_j] + b_i f1[b_j]]] // Ad[γ[g, i, j]],
  6, R[i_, j_][x_] := x // Ad[a[1, i, j]] // Ad[β[f0[b_j] + b_i f1[b_j]]] // Ad[γ[g[b_i, b_j], i, j]] //
    Ad[γa[-(ⅇ^-b_1 (b_1 - ⅇ^b_1 b_1 + b_2 - ⅇ^b_1 b_2 + ⅇ^b_1 b_1 b_2))/(b_1^3 b_2), i, j, i, j]]
 ];
VerifyR3[expr_] := Module[{lhs, rhs}, {
   lhs = expr // R[1, 2] // R[1, 3] // R[2, 3] // LSimp;
   rhs = expr // R[2, 3] // R[1, 3] // R[1, 2] // LSimp;
   LSimp[lhs - rhs] == 0
  }]
```

Verifying R3

```
VerifyR3 /@ {γ[f, 1, 4], γ[f, 2, 4], γ[f, 3, 4], γ[f, 4, 1], γ[f, 4, 2], γ[f, 4, 3]}
```

```
{{True}, {True}, {True}, {True}, {True}, {True}}
```

```
VerifyR3 /@ {a[f[b_1, b_2, b_3, b_4], 1, 4], a[f, 2, 4]}
```

```
{{True}, {γ[(-1 + ⅇ^b_1) f, 2, 3, 4] + γa[((-1 + ⅇ^b_1) f)/b_1, 2, 4, 1, 3] +
    γa[(f - ⅇ^b_1 f)/b_1, 2, 3, 1, 4] + γa[(ⅇ^-b_2 (-1 + ⅇ^b_1) (-1 + ⅇ^b_2) f (-2 + b_2))/b_1^2, 1, 3, 1, 4] +
    γa[-(ⅇ^-b_2 (-1 + ⅇ^b_1) (-1 + ⅇ^b_2) f (-2 + b_2))/(b_1 b_2), 1, 3, 2, 4] == 0}}
```

$$\mathbf{VerifyR3\ /@\ \{a[f[b_1,\ b_2,\ b_3,\ b_4],\ 1,\ 4],\ a[f[b_1,\ b_2,\ b_3,\ b_4],\ 2,\ 4],\ a[f[b_1,\ b_2,\ b_3,\ b_4],\ 3,\ 4]\}}$$

$$\Big\{\{\text{True}\},\ \Big\{\gamma\Big[\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4],2,3,4\Big]+$$

$$\gamma a\Big[-\frac{\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]}{b_1},2,3,1,4\Big]+\gamma a\Big[\frac{\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]}{b_1},2,4,1,3\Big]+$$

$$\gamma a\Big[\frac{\mathbb{e}^{-b_2}\left(-1+\mathbb{e}^{b_1}\right)\left(-1+\mathbb{e}^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(-2+b_2)}{b_1^2},1,3,1,4\Big]+$$

$$\gamma a\Big[-\frac{\mathbb{e}^{-b_2}\left(-1+\mathbb{e}^{b_1}\right)\left(-1+\mathbb{e}^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(-2+b_2)}{b_1\,b_2},1,3,2,4\Big]==0\Big\},$$

$$\Big\{\gamma\Big[-\frac{1}{b_2}\mathbb{e}^{b_1}\left(-1+\mathbb{e}^{b_2}\right) f[b_1,b_2,b_3,b_4]\,b_3\,(f0'[b_2]-f0'[b_3]+b_1\,(f1'[b_2]-f1'[b_3])),2,4\Big]+\gamma\Big[-\frac{1}{b_1}$$

$$f[b_1,b_2,b_3,b_4]\,b_3\,\Big(\left(-1+\mathbb{e}^{b_1}\right) f1[b_2]+\mathbb{e}^{b_1}\left(-1+\mathbb{e}^{b_2}\right) g[b_1,b_2]\,b_1+\mathbb{e}^{b_1}\,g[b_1,b_3]\,b_1-\mathbb{e}^{b_1+b_2}\,g[b_1,b_3]\,b_1-$$

$$\mathbb{e}^{b_2}\,g[b_2,b_3]\,b_2+\mathbb{e}^{b_1+b_2}\,g[b_2,b_3]\,b_2-\mathbb{e}^{b_2}\,f0'[b_3]+\mathbb{e}^{b_1+b_2}\,f0'[b_3]-\mathbb{e}^{b_2}\,b_2\,f1'[b_3]+\mathbb{e}^{b_1+b_2}\,b_2\,f1'[b_3]\Big),$$

$$1,4\Big]+\gamma\Big[\frac{\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\,b_3}{b_1},1,2,4\Big]+\gamma\Big[-\frac{1}{b_1^3\,b_2^2}\,\mathbb{e}^{-b_1}\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]$$

$$\Big(-\mathbb{e}^{b_2}\left(-1+\mathbb{e}^{b_1}\right) b_2^3\,b_3+\mathbb{e}^{b_2}\,b_1\,b_2^2\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right) b_3+\mathbb{e}^{b_1}\,b_1^3\left(b_2\left(-1+\mathbb{e}^{b_2}-b_3\right)+\left(-1+\mathbb{e}^{b_2}\right) b_3\right)\Big),2,3,4\Big]+$$

$$\gamma\Big[\frac{1}{b_1^4\,b_2}\,\mathbb{e}^{-b_1-b_2}\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\,\Big(\mathbb{e}^{2\,b_2}\left(-1+\mathbb{e}^{b_1}\right) b_2^3\,b_3-\mathbb{e}^{2\,b_2}\,b_1\,b_2^2\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right) b_3+$$

$$\mathbb{e}^{b_1}\,b_1^3\left(\left(1+\mathbb{e}^{b_2}-2\,\mathbb{e}^{2\,b_2}\right) b_3+b_2\left(-1+\mathbb{e}^{b_2}+\left(-1+\mathbb{e}^{b_2}+\mathbb{e}^{2\,b_2}\right) b_3\right)\right)\Big),1,3,4\Big]+\gamma a\Big[\frac{1}{b_1^4\,b_2}$$

$$\mathbb{e}^{-b_1}\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\left(\mathbb{e}^{b_2}\left(-1+\mathbb{e}^{b_1}\right) b_2^3-\mathbb{e}^{b_2}\,b_1\,b_2^2\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right)+\mathbb{e}^{b_1}\,b_1^3\left(2-2\,\mathbb{e}^{b_2}+\mathbb{e}^{b_2}\,b_2\right)\right),$$

$$1,3,3,4\Big]+\gamma a\Big[\frac{\left(-1+\mathbb{e}^{b_1}\right)\left(-1+\mathbb{e}^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(-2+b_2)\,b_3}{b_1^2\,b_2},1,2,1,4\Big]+$$

$$\gamma a\Big[-\frac{\mathbb{e}^{-b_2}\left(-1+\mathbb{e}^{b_1}\right)\left(-1+\mathbb{e}^{2\,b_2}\right) f[b_1,b_2,b_3,b_4]\,(-2+b_2)\,b_3}{b_1^2\,b_2},1,3,1,4\Big]+$$

$$\gamma a\Big[-\frac{\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\left(2-2\,\mathbb{e}^{b_2}+\mathbb{e}^{b_2}\,b_2\right) b_3}{b_1\,b_2^2},1,2,2,4\Big]+$$

$$\gamma a\Big[\frac{\mathbb{e}^{-b_2}\left(-1+\mathbb{e}^{b_1}\right)\left(-1+\mathbb{e}^{b_2}\right) f[b_1,b_2,b_3,b_4]\,(-b_3+b_2\,(1+b_3))}{b_1\,b_2^2},1,4,2,3\Big]+$$

$$\gamma a\Big[\frac{1}{b_1^4\,b_2^2}\,\mathbb{e}^{-b_1}\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\,\Big(-\mathbb{e}^{b_2}\left(-1+\mathbb{e}^{b_1}\right) b_2^3\,b_3+\mathbb{e}^{b_2}\,b_1\,b_2^2\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right) b_3+$$

$$\mathbb{e}^{b_1}\,b_1^3\left(b_2\left(-1+\mathbb{e}^{b_2}-b_3\right)+\left(-1+\mathbb{e}^{b_2}\right) b_3\right)\Big),2,3,1,4\Big]+\gamma a\Big[-\frac{1}{b_1}\mathbb{e}^{b_1}\left(-1+\mathbb{e}^{-b_1}\right) f[b_1,b_2,b_3,b_4]$$

$$\left(-\frac{\mathbb{e}^{-b_1+b_2}\left(-\left(-1+\mathbb{e}^{b_1}\right) b_2+b_1\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right)\right) b_3}{b_1^3}+\frac{b_2\,b_3-\left(-1+\mathbb{e}^{b_2}\right)(b_2+b_3)}{b_2^2}\right),2,4,1,3\Big]+$$

$$\gamma a\Big[-\frac{1}{b_1^4\,b_2^2}\,\mathbb{e}^{-b_1-b_2}\left(-1+\mathbb{e}^{b_1}\right) f[b_1,b_2,b_3,b_4]\,\Big(2\,\mathbb{e}^{2\,b_2}\left(-1+\mathbb{e}^{b_1}\right) b_2^3\,b_3-2\,\mathbb{e}^{2\,b_2}\,b_1\,b_2^2\left(1-\mathbb{e}^{b_1}+\mathbb{e}^{b_1}\,b_2\right) b_3+$$

$$\mathbb{e}^{b_1}\,b_1^3\left(-\left(1-3\,\mathbb{e}^{b_2}+2\,\mathbb{e}^{2\,b_2}\right) b_3+b_2\left(-1+\mathbb{e}^{b_2}+\mathbb{e}^{b_2}\left(1+\mathbb{e}^{b_2}\right) b_3\right)\right)\Big),1,3,2,4\Big]==0\Big\}\Big\}$$