Cheat Sheet OneCo

http://drorbn.net/AcademicPensieve/2015-05/
initiated 14/4/15; modified 9/6/15, 2:50am; continues 2015-04, continued 2015-06

Deriving Gassner. $\mathcal{L}^{2Dw}$ is $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo locality, $[a_{ij},a_{ik}]=0$, $[a_{ik},a_{jk}]=-[a_{ij},a_{jk}]=b_ja_{ik}-b_ia_{jk}$, and (mod $\langle a_{ii}\rangle$) $[a_{ij},a_{ji}]=b_ia_{ji}-b_ja_{ij}$. Acts on $V=\mathbb{Q}[\![b_i]\!]\langle x_i=a_{i\infty}\rangle$ by $[a_{ij},x_i]=0$, $[a_{ij},x_j]=b_ix_j-b_jx_i$. Hence $e^{\operatorname{ad}a_{ij}}x_i=x_i$, $e^{\operatorname{ad}a_{ij}}x_j=e^{b_i}x_j+\frac{b_j}{b_i}(1-e^{b_i})x_i$. Renaming $y_i=x_i/b_i$, $t_i=e^{b_i}$, get $[e^{\operatorname{ad}a_{ij}}]_{y_i,y_j}=\begin{pmatrix}1 & 1-t_i\\ 0 & t_i\end{pmatrix}$.

The $\mathcal{L}^{2Dw}$ Adjoint representation. $e^{\operatorname{ad}a_{ij}}$ acts by

$$a_{kl}\mapsto a_{kl},\qquad a_{ik}\mapsto a_{ik},\qquad a_{kj}\mapsto e^{-b_i}a_{kj}+\frac{b_k}{b_i}(1-e^{-b_i})a_{ij},$$

$$a_{ki}\mapsto a_{ki}+(1-e^{-b_i})a_{kj}+b_k\frac{e^{-b_i}-1}{b_i}a_{ij},$$

$$a_{jk}\mapsto e^{b_i}a_{jk}+\frac{b_j}{b_i}(1-e^{b_i})a_{ik},\qquad a_{ji}\mapsto e^{b_i}a_{ji}+\frac{b_j}{b_i}(1-e^{b_i})a_{ij}.$$

Adjoint Gassner. Renaming $\alpha_{ij}=a_{ij}/b_i$ and $t_i=e^{b_i}$, get

$$\alpha_{kj}\mapsto t_i^{-1}\alpha_{kj}+(1-t_i^{-1})\alpha_{ij},$$

$$\alpha_{ki}\mapsto a_{ki}+(1-t_i^{-1})\alpha_{kj}+(t_i^{-1}-1)\alpha_{ij}$$

$$\alpha_{jk}\mapsto t_i\alpha_{jk}+(1-t_i)\alpha_{ik},\qquad \alpha_{ji}\mapsto t_i\alpha_{ji}+(1-t_i)\alpha_{ij}.$$

Implementation/verification: pensieve://2015-04/nb/ZeroCo.pdf.
Interpretation: π_T-Artin?

2Dv. b: bracket trace; c: cobracket trace; $\langle b,c\rangle=\delta\in\{0,1\}$; $\deg b_i=\deg c_j=\deg a_{ij}=\deg\delta=1$.
$\mathcal{A}^{2Dv}$ is $\mathbb{Q}[\![\delta]\!]FA(b_i,c_j,a_{ij})$ (so $\mathcal{L}^v=\{f+f^{ij}a_{ij}\}$) modulo locality,

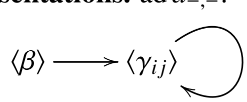
tt. $[a_{jk},a_{jl}]=c_la_{jk}-c_ka_{jl}=:\gamma_{jkl}$, (note $\gamma_{j\{kl\}}=0$)
hh. $[a_{jk},a_{ik}]=b_ia_{jk}-b_ja_{ik}$,
th. $[a_{jk},a_{ij}]=b_ja_{ik}-b_ia_{jk}+\gamma_{ijk}$,
⇆. $[a_{ij},a_{ji}]=$?,
ab, ac. $[a_{ij},b_i]=-[a_{ij},b_j]=-[a_{ij},c_i]=[a_{ij},c_j]=\delta a_{ij}-b_ic_j=:\gamma_{ij}$,
bc. $[b_i,c_j]=0$.

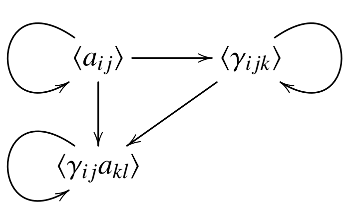
So $a_{ij}f=f^\delta a_{ij}-\dfrac{b_ic_j}{\delta}(f^\delta-f)$, $[a_{ij},f]=(f^\delta-f)\left(a_{ij}-\dfrac{b_ic_j}{\delta}\right)$,
with $f^\delta:=f/\!/\begin{pmatrix}b_i\to b_i+\delta & b_j\to b_j-\delta\\ c_i\to c_i-\delta & c_j\to c_j+\delta\end{pmatrix}$.

The Ascending Algebra $\mathcal{A}_+^{2Dv}$. Same but with only a_{ij}, $i<j$.
The OneCo Sub-Quotient is $\langle a_{ij}\rangle$ modulo $\delta^2=\delta c_i=c_jc_k=0$, so $\mathcal{L}^{1co}$ is (coefficient functions non-central, in $\mathbb{Q}[\![b_i]\!]$)

$$\left\{f^{ij}a_{ij}+f^{ijk}\gamma_{ijk}+f^{ijkl}\gamma_{ij}a_{kl}\right\}\Big/\begin{matrix}(b_i\gamma_{ijk}=\gamma_{ij}a_{ik}-\gamma_{ik}a_{ij})\\ \text{(is there a residual 4T?)}\end{matrix}\quad.$$

Representations. ad $a_{-,-}$:





• $\langle\gamma_{ij}\rangle$ is ...

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B[a[f_, j_, k_], β[g_]] := γ[f (∂_{b_j} g - ∂_{b_k} g), j, k] // LSimp;

B[a[j_, k_], a[j_, l_]] /; DQ[j, k, l] := γ[1, j, k, l] // LSimp;
B[a[j_, k_], a[i_, k_]] /; DQ[i, j, k] :=
  a[b_i, j, k] - a[b_j, i, k] // LSimp;
B[a[j_, k_], a[i_, j_]] /; DQ[i, j, k] :=
  a[b_j, i, k] - a[b_i, j, k] + γ[1, i, j, k] // LSimp;
B[a[j_, k_], a[k_, l_]] /; DQ[j, k, l] := -B[a[k, l], a[j, k]];
B[a[j_, k_], a[l_, m_]] /; DQ[j, k, l, m] := 0;


B[a[f_, j_, k_], γ[g_, j_, l_]] /; DQ[j, k, l] := 0;
B[a[f_, j_, k_], γ[g_, i_, k_]] /; DQ[i, j, k] :=
  - γ[b_j f g, i, k] // LSimp;
B[a[f_, j_, k_], γ[g_, i_, j_]] /; DQ[i, j, k] := γ[b_j f g, i, k] // LSimp;
B[a[f_, j_, k_], γ[g_, k_, l_]] /; DQ[j, k, l] :=
  γ[b_j f g, k, l] - γ[b_k f g, j, l] // LSimp;
B[a[f_, j_, k_], γ[g_, l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, j_, k_]] := γ[-b_j f g, j, k] // LSimp;


B[a[f_, j_, k_], γ[g_, j_, l_, m_]] /; DQ[j, k, l, m] := 0;
B[a[f_, j_, k_], γ[g_, i_, j_, l_]] /; DQ[i, j, k, l] :=
  γ[b_j f g, i, k, l] + γa[f g, i, l, j, k] // LSimp;
B[a[f_, l_, k_], γ[g_, i_, j_, l_]] /; DQ[i, j, k, l] :=
  γ[-b_l f g, i, k, j] + γa[-f g, i, j, l, k] // LSimp;
B[a[f_, j_, k_], γ[g_, k_, l_, m_]] /; DQ[j, k, l, m] :=
  γ[-b_k f g, j, l, m] + γ[b_j f g, k, l, m] // LSimp;
B[a[f_, j_, k_], γ[g_, n_, i_, k_]] /; DQ[n, i, j, k] :=
  γ[-b_j f g, n, i, k] + γa[f g, n, i, j, k] // LSimp;
B[a[f_, j_, i_], γ[g_, n_, i_, k_]] /; DQ[n, i, j, k] :=
  γ[b_j f g, n, k, i] + γa[-f g, n, k, j, i] // LSimp;
B[a[f_, j_, k_], γ[g_, j_, k_, l_]] /; DQ[j, k, l] :=
  γa[-f g, j, k, j, l] // LSimp;
B[a[f_, j_, l_], γ[g_, j_, k_, l_]] /; DQ[j, k, l] :=
  γa[f g, j, l, j, k] // LSimp;
B[a[f_, j_, k_], γ[g_, i_, j_, k_]] /; DQ[i, j, k] :=
  γ[-b_j f g, i, j, k] + γa[f g, i, j, j, k] + γa[f g, i, k, j, k] // LSimp;
B[a[f_, k_, j_], γ[g_, i_, j_, k_]] /; DQ[i, j, k] :=
  γ[b_k f g, i, k, j] + γa[-f g, i, k, k, j] + γa[-f g, i, j, k, j] // LSimp;
B[a[f_, i_, j_], γ[g_, k_, l_, m_]] /; DQ[i, j, k, l, m] := 0;
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Print[# -> Ad[a[t, j, k]][#]] & /@ tests;
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$$\beta[f[b_j, b_k]] \to \beta[f[b_j, b_k]] + \gamma\left[\frac{(-1+e^{-t b_j})\,(f^{(0,1)}[b_j,b_k]-f^{(1,0)}[b_j,b_k])}{b_j}, j, k\right]$$

$$\gamma[1, i, j] \to \gamma[1, i, j] + \gamma[1 - e^{-t b_j}, i, k]$$

$$\gamma[1, i, k] \to \gamma[e^{-t b_j}, i, k]$$

$$\gamma[1, i, l] \to \gamma[1, i, l]$$

$$\gamma[1, j, k] \to \gamma[e^{-t b_j}, j, k]$$

$$\gamma[1, j, l] \to \gamma[1, j, l]$$

$$\gamma[1, k, l] \to \gamma[e^{t b_j}, k, l] + \gamma\left[\frac{(1-e^{t b_j})\,b_k}{b_j}, j, l\right]$$

$$\gamma[1, j, l, m] \to \gamma[1, j, l, m]$$

$$\gamma[1, i, j, l] \to \gamma[1, i, j, l] + \gamma[1 - e^{-t b_j}, i, k, l] + \gamma a\left[\frac{1-e^{-t b_j}}{b_j}, i, l, j, k\right]$$

$$\gamma[1, i, k, l] \to \gamma[e^{-t b_j}, i, k, l] + \gamma a\left[\frac{-1+e^{-t b_j}}{b_j}, i, l, j, k\right]$$

$$\gamma[1, k, l, m] \to \gamma[e^{t b_j}, k, l, m] + \gamma\left[\frac{(1-e^{t b_j})\,b_k}{b_j}, j, l, m\right]$$

$$\gamma[1, j, k, l] \to \gamma[1, j, k, l] + \gamma a\left[\frac{-1+e^{-t b_j}}{b_j}, j, k, j, l\right]$$

$$\gamma[1, i, j, k] \to \gamma[e^{-t b_j}, i, j, k] + \gamma a\left[\frac{1-e^{-t b_j}}{b_j}, i, j, j, k\right] + \gamma a\left[\frac{1-e^{-t b_j}}{b_j}, i, k, j, k\right]$$

$$a[1, j, k] \to a[1, j, k]$$

$$a[1, j, l] \to a[1, j, l] + \gamma[t, j, k, l] + \gamma a\left[\frac{1-e^{-t b_j}-t b_j}{b_j^2}, j, k, j, l\right]$$

$$a[1, i, k] \to a[e^{-t b_j}, i, k] + a\left[\frac{(1-e^{-t b_j})\,b_i}{b_j}, j, k\right] + \gamma a\left[\frac{e^{-2 t b_j}\,b_i\,(1-e^{2 t b_j}+2\,e^{t b_j}\,t\,b_j)}{b_j^3}, j, k, j, k\right] + \gamma a\left[\frac{e^{-2 t b_j}\,(-1+e^{t b_j}\,(1-t\,b_j))}{b_j^2}, j, k, i, k\right]$$

$$a[1, i, j] \to a[1, i, j] + a[1 - e^{-t b_j}, i, k] + a\left[\frac{(-1+e^{-t b_j})\,b_i}{b_j}, j, k\right] + \gamma\left[\frac{1-e^{-t b_j}}{b_j}, i, j, k\right] + \gamma a\left[\frac{-1+e^{-t b_j}+t\,b_j}{b_j^2}, i, j, j, k\right] + \gamma a\left[\frac{-1+e^{-t b_j}+t\,b_j}{b_j^2}, i, k, j, k\right] + \gamma a\left[\frac{b_i\,(1-e^{-2 t b_j}-2\,e^{-t b_j}\,t\,b_j)}{b_j^3}, j, k, j, k\right] + \gamma a\left[\frac{e^{-2 t b_j}\,(1+e^{t b_j}\,(-1+t\,b_j))}{b_j^2}, j, k, i, k\right]$$

$$a[1, k, l] \to a[e^{t b_j}, k, l] + a\left[\frac{(1-e^{t b_j})\,b_k}{b_j}, j, l\right] + \gamma\left[\frac{t\,b_j\,b_k+(1-e^{t b_j})\,(b_j+b_k)}{b_j^2}, j, k, l\right] + \gamma a\left[\frac{1+e^{t b_j}\,(-1+t\,b_j)}{b_j^2}, j, k, k, l\right] + \gamma a\left[\frac{-2\,b_j+e^{-t b_j}\,b_j-2\,b_k-t\,b_j\,b_k+e^{t b_j}\,(b_j+2\,b_k-t\,b_j\,b_k)}{b_j^3}, j, k, j, l\right]$$

$$a[1, l, m] \to a[1, l, m]$$

$$\gamma a[1, j, k, j, l] \to \gamma a[e^{-t b_j}, j, k, j, l]$$

To do. • Perhaps I should find a way to highlight the fact that v is a perturbation of w. • Position FiC. • Position the 2D Lie bialgebras. • Is there a meaningful $a_{ij} \to a_{ij}/b_i$ (etc) renormalization? • Add: diagrammatic interpretations of b_i, c_j, γ_{ij}, γ_{ijk}.

Recycling.

Models. • In $[x,y]=\delta x$, $xf(y)=f(y+\delta)x$. If $\delta^2=0$, $[x,f(y)]=\delta f'(y)x$.

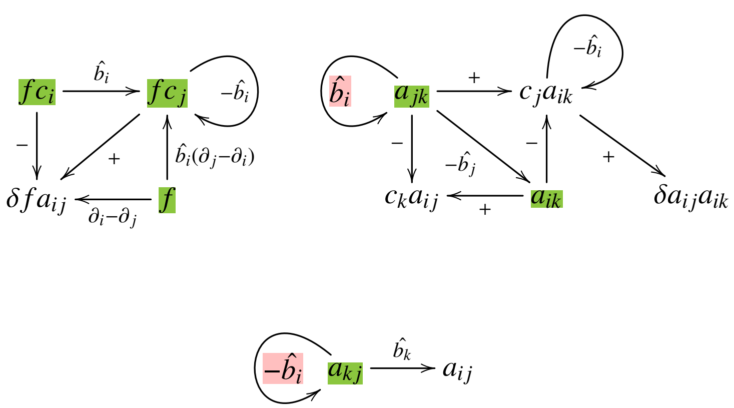
• In $[x,y]=\delta x+z^2$, $xf(y)=f(y+\delta)x+\frac{z^2}{\delta}(f(y+\delta)-f(y))$. If $\delta^2=0$, $[x,f(y)]=\delta f'(y)x+z^2f'(y)$.

• If $S_n\coloneqq\sum_{k=0}^{n-1}A^kCB^{n-1-k}$ then $AS_n-S_nB=A^nC-CB^n$ so $S_n=(L_A-R_B)^{-1}(A^nC-CB^n)$.

• If $\psi(x)=\sum_{n\geq 0}a_nx^n$ then $\sum_{n\geq 0}a_n\sum_{k=0}^{n-1}b^n(-b)^{n-1-k}=(\psi(b)-\psi(-b))/2b$.

The primitivity condition. $\ker\left(f+f^{ij}a_{ij}\mapsto\delta f+f^{ij}b_ic_j\right)$. (Ignoring multiple arrows).

State Diagrams. $\operatorname{ad} a_{ij}$ yields green: roots. pink: wrong.



so with $\phi_0\coloneqq\phi(0)$, $\phi_1\coloneqq\phi'_0$, and $\phi_\downarrow(x)\coloneqq(\phi(x)-\phi_0)/x$, $\phi(\operatorname{ad}a_{ij})$ is

$$fc_i\mapsto\phi_0fc_i+(b_i\phi_{\downarrow\downarrow}(-b_i)-\phi_1)\delta fa_{ij}+b_i\phi_\downarrow(-b_i)fc_j$$
$$fc_j\mapsto\phi(-b_i)fc_j+\phi_\downarrow(-b_i)\delta fa_{ij}$$
$$f\mapsto\phi_0f+b_i\phi_\downarrow(-b_i)(\partial_jf-\partial_if)c_j+(b_i\phi_{\downarrow\downarrow}(-b_i)-\phi_1)(\partial_jf-\partial_if)\delta a_{ij}$$
$\delta a_{..}\mapsto$ as in Adjoint Gassner
$$a_{ik}\mapsto\phi_0a_{ik}+\phi_1c_ka_{ij}-\phi_\downarrow(-b_i)c_ja_{ik}-\phi_{\downarrow\downarrow}(-b_i)\delta a_{ij}a_{ik}$$
$$a_{jk}\mapsto\phi(b_i)a_{jk}-(\phi_\downarrow(b_i)+b_j\phi_{\downarrow\downarrow}(b_i))c_ka_{ij}-b_j\phi_\downarrow(b_i)a_{ik}+\frac{\phi(b_i)-\phi(-b_i)+b_j(\phi_\downarrow(b_i)-\phi_\downarrow(-b_i))}{2b_i}c_ja_{ik}+\frac{\phi_\downarrow(b_i)-\phi_\downarrow(-b_i)+b_j(\phi_{\downarrow\downarrow}(b_i)-\phi_{\downarrow\downarrow}(-b_i))}{2b_i}\delta a_{ij}a_{ik}$$
$$a_{kj}\mapsto$$
$$a_{ij}\mapsto a_{ij}$$

Then $[a_{ij},f]=(\partial_if-\partial_jf)\gamma_{ij}$ and

γb. $[\gamma_{ij},b_l]=0$ and $[\gamma_{ijk},b_l]=0$ incl. $l\in\{i,j,k\}$,

ttγ. $[a_{jk},\gamma_{jl}]=0$,

hhγ. $[a_{jk},\gamma_{ik}]=-b_j\gamma_{ik}$,

thγ. $[a_{jk},\gamma_{ij}]=b_j\gamma_{ik}$,

htγ. $[a_{jk},\gamma_{kl}]=b_j\gamma_{kl}-b_k\gamma_{jl}$,

ttγ_3. $[a_{jk},\gamma_{jlm}]=0$,

thγ_3. $[a_{jk},\gamma_{ijl}]=b_j\gamma_{ikl}+\gamma_{il}a_{jk}$,

htγ_3. $[a_{jk},\gamma_{klm}]=b_k\gamma_{jkl}+b_j\gamma_{klm}$,

hhγ_3. $[a_{jk},\gamma_{nik}]=-b_j\gamma_{nik}+\gamma_{ni}a_{jk}$,

$[a_{jk},\gamma_{jkl}]=-\gamma_{jk}a_{jl}$,

$[a_{jk},\gamma_{ijk}]=-b_j\gamma_{ijk}+\gamma_{ij}a_{jk}+\gamma_{ik}a_{jk}$.